Eigenvalue distributions on a single ring

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Abstract

In 1965 J. Ginibre introduced an ensemble of random matrices with no symmetry conditions imposed as the mathematical counterpart to hermitian random matrix theory. In his original paper he treats the case of matrices with i.i.d. normally distributed real, complex or quaternion entries. Since then, mainly due to interest from applications, the development of non-hermitian random matrix theory has further evolved, though the eigenvalue statistics of non-hermitian random matrices are far from being as thoroughly understood as their hermitian counterpart. A characteristic of non-hermitian random matrices are eigenvalue distributions in the complex plane. Real asymmetric random matrices have the additional caveat of having real and complex eigenvalues and thus are technically more challenging. In the following work a new three-fold family of non-hermitian random matrices is introduced via a quadratization procedure. As a consequence the entries of these matrices are highly dependent. For all three ensembles the joint eigenvalue probability density functions and eigenvalue correlations are derived for $\beta = 1, 2$. In the limit of large matrix dimensions a classification of eigenvalue correlation functions for different asymptotic regimes is undertaken. In tune with the title of this work for all three ensembles there exists an asymptotic regime, in which the eigenvalues are supported on an annulus around the origin. Thus the induced family of non-hermitian random matrix ensembles serves as an example, for ensembles of the Feinberg-Zee type with logarithmic potential.

Statement of originality

This thesis is submitted to the University of London for the degree of Doctor of Philosophy. It is my own work and only contains results with which I have been directly involved. Section 2.1 is taken from [FBK$^+$11] and is based on an idea of H.-J. Sommers. Section 3.4.2 is taken from [FF11] and is based on work by P. Forrester. Any further results of others have been fully acknowledged.

Signed:

Dated:

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Chapter 1

Introduction

1.1 Thesis outline

In the following work we introduce a new family of non-hermitian random matrix ensembles. This family consist of three ensembles with either real or complex matrix entries and are obtained through an inducing procedure described in chapter 2. We start with a short historical overview of random matrix theory and its application, followed by an introduction to non-hermitian random matrix theory. In chapter 2 the inducing procedure is described, which further on, is used to generate the family of induced ensembles. In addition chapter 3 deals with the induced family of non-hermitian random matrices with complex entries: solving the induced complex Ginibre ensemble and then the complex induced spherical and Jacobi ensemble. The section on the induced complex Ginibre ensemble highlights the general methods used in the context of non-hermitian random matrix ensembles with complex entries. A short paragraph at the end of chapter 2 introduces the application of complex non-hermitian random matrix theory to the theory of the two-dimensional one-component plasma. Furthermore chapter 4 deals with the induced family of non-hermitian ensembles with real matrix entries. Again the real induced Ginibre ensemble is solved first and general methods necessary for dealing with non-hermitian random matrices with real entries are introduced. These methods are then applied to the real induced spherical and the real induced Jacobi ensemble. We end with some concluding remarks in chapter 5.

1.2 Random matrix theory

1.2.1 Historical overview

Very simply put a random matrix is a matrix whose entries are random variables. A short and very readable description of random matrices can be found in [Dia05]. Of particular interest in random matrix theory are the statistical properties of the eigenvalues of large random matrices. Random matrices were first studied in the 1920's in mathematical statistics [Wis28] by Wishart in the context of random covariance matrices, but did not attract much attention at the time. However in the 1950's they were introduced to the field of nuclear physics by Wigner, using the spacing distribution between two eigenvalues in order to model the behavior of energy levels of excitations of heavy nuclei [Wig55b, Wig55a, Wig57a]. A historical overview of random matrix theory and its development can be found in [FSV03]. Since then random matrices have appeared in a diverse variety of different context with applications in quantum mechanics, statistical physics, finance, genetics, wireless networks, number theory [MS05] and graph theory just to name a few. Standard works on random matrix theory include [AGZ10, For10b, Meh04, PS11]. A recent overview of random matrix theory was undertaken in [ABDF11].

The ubiquity of random matrices and random matrix distributions can be explained by one of the main features of random matrix theory. In the limit of large matrix dimension certain statistical properties of the eigenvalues of random matrices do not depend on the precise distribution of matrix elements, but only on some invariant properties of the underlying random matrix ensemble. This phenomena is referred to as universality in random matrix theory.

As the development of random matrix theory was largely driven by applications, the main focus of the field lay in the study of hermitian random matrix ensembles. These are random matrices with symmetry conditions imposed, such that the distribution of eigenvalues is concentrated on the real line. Traditionally one distinguishes three types of random matrix ensembles denoted by the Dyson index β, where $\beta = 1, 2, 4$ refers to ensembles with complex, real or quaternion entries respectively [Dys62b, Dys62c, Dys62d].

The simplest and probably most studied hermitian random matrix ensemble is the Gaussian Unitary Ensemble (GUE). Using it as an example will give us the opportunity to highlight some concepts and ideas of RMT.

Definition 1.2.1. *The Gaussian Unitary Ensemble (GUE) is the space of hermitian matrices $X = (x_{jk})_{j,k=1}^{N} \in \mathbb{C}^{N\times N}$, whose elements are independent normal random variables with probability density function:*

$$p(x_{jk}) = \frac{2}{\pi}\,\mathrm{e}^{-2|x_{jk}|^2} = \frac{2}{\pi}\,\mathrm{e}^{-2\,\mathrm{Re}(x_{jk})^2 - 2\,\mathrm{Im}(x_{jk})^2}, \quad \textit{for} \quad j<k \tag{1.2.1}$$

$$p(x_{jj}) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-x_{jj}^2}, \quad \textit{for} \quad j=k. \tag{1.2.2}$$

Note that the joint probability density function (jpdf) of the entries of X factorizes, allowing us to write:

$$P(X) = \frac{2^{\frac{1}{2}N(N-1)}}{\pi^{\frac{1}{2}N^2}}\,\mathrm{e}^{-\operatorname{tr}(X^2)}\,. \tag{1.2.3}$$

The eigenvalues of X are with probability one distinct and thus can be ordered, which allows us to view them as random variables. Their jpdf can be obtained by changing variables from the matrix entries of X to the eigenvalues of X and some auxiliary variables using the spectral decomposition. Integrating out the latter yields:

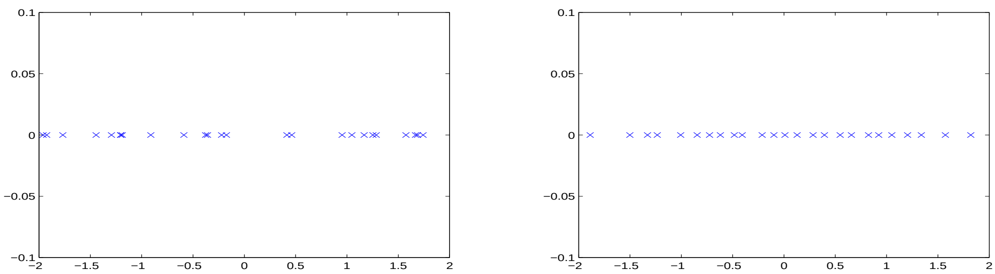
$$p(\lambda_1,\ldots,\lambda_N) = c\prod_{1\le j<k\le N}|\lambda_j-\lambda_k|^2\prod_{j=1}^{N}\mathrm{e}^{-|\lambda_j|^2}\,. \tag{1.2.4}$$

Note that the Vandermonde determinant $\Delta = \prod_{1\le j<k\le N}|\lambda_j-\lambda_k|^2$ is actually the Jacobian of this change of variables. Moreover the factor Δ implies that the probability of two eigenvalues lying in close vicinity to each other is small. This phenomena is referred to in random matrix theory as eigenvalue repulsion. Thus eigenvalues of a hermitian random matrix exhibit fundamentally different behavior, than for example, points of a Poisson process on the real line, as shown in figure 1.1.

Another important quantity of interest in random matrix theory is the so-called mean eigenvalue density $\rho_N(\lambda)$. Integrating the mean eigenvalue density over a particular set, gives the expected number of eigenvalues that fall into this set. Wigner's most famous result relates to the mean eigenvalue density in the limit of large matrix dimensions and is known as Wigner's semi-circle law [Wig57b]:

$$\rho_N(\lambda) = \frac{1}{\pi N}\sqrt{2N-\lambda^2}. \tag{1.2.5}$$

Another breakthrough for hermitian RMT came in 1970 when Dyson [Dys62a]



(a) 25 points uniformly distributed on the real interval (-2,2)

(b) 25 eigenvalues of a GUE matrix

Figure 1.1: Point distributions on the line

succeeded in expressing the correlation functions of a different random matrix ensemble for $\beta = 2$ in terms of a determinant. Further progress was achieved by Dyson and Mehta [Meh04], who succeeded in using skew-orthogonal polynomials in order to express the correlation functions for $\beta = 1, 4$ in terms of quaternion determinants.

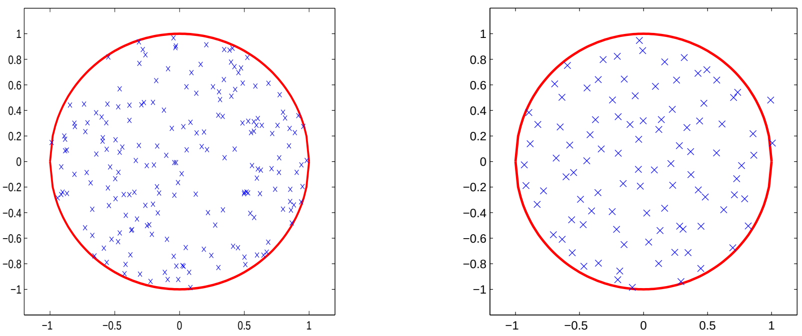
Since then significant advances were made in the study of hermitian random matrix ensembles. As a result their statistical eigenvalue behavior in the limit of large matrix dimensions is now relatively well understood. A good overview on universality for hermitian matrices can be found in [TW00]. A more recent overview is given in [Kui11].

1.2.2 Non-hermitian random matrix theory

Non-hermitian random matrices were first introduced by Jean Ginibre in 1965, as an extension to the mathematical theory of hermitian random matrices [Gin65]. Their main feature being eigenvalues distributions in the complex plane. In his original paper Ginibre derives the joint eigenvalue probability density function of matrices with i.i.d. normally distributed complex, quaternion or real entries. These Ginibre ensembles are sometimes denoted in the literature [Sin07] as GinOE, GinUE and GinSE, respectively. The letter U, O and S stands for the orthogonal, unitary and symplectic symmetry class. The case GinOE of real asymmetric matrices proved to be the hardest and Ginibre studied only the special case that all eigenvalues are real. It took another 25 years for Lehmann and Sommers [LS91] and Edelman [Ede97] to derive the complete distribution of eigenvalues for the real Ginibre ensemble. Further difficulty arose in the computation of the eigenvalue correlation functions. In 2007 Akemann and Kanzieper succeeded in expressing the complex-complex correlation functions as Pfaffians [AK07], whereas Sinclair presented a method for averaging over the real Ginibre ensemble in terms of Pfaffians [Sin07]. Finally, Forrester and Nagao were able to determine the real-real as well as the complex-complex correlation functions as Pfaffians using the method of skew-orthogonal polynomials [FN07, FN08], while Borodin and Sinclair gave the real-complex correlation in addition to a thorough asymptotic analysis [BS09]. Simultaneously and independently, Sommers [Som07] and Sommers and Wieczorek [SW08] derived the complex-complex, real-real, and complex-real eigenvalue correlation functions via free-fermion diagram expansion. A general review on non–Hermitian random matrices can be found in [FS03], while a recent overview on the Ginibre ensembles is provided in [KS09].

Even though non-hermitian random matrices were initially introduced purely because of mathematical curiosity, more and more applications of complex non-hermitian and real asymmetric random matrices were discovered. They appear in the study of dissipative quantum maps [GHS88], scattering in chaotic quantum systems [FS11], growth processes [Joh06], the stability of complex biological networks [May72], neural networks [SCSS], directed quantum chaos [Efe97], random operations in quantum information theory [BCSŻ09, FF11] and others. One prominent application is the two-dimensional one-component plasma. At inverse temperature $\beta = 2$ the Boltzmann factor of a one-component plasma on certain two-dimensional surfaces coincides with the eigenvalue joint probability density function of specific non-hermitian random matrix ensembles, also at $\beta = 2$. Examples can be found in [FN07]. Furthermore real asymmetric matrices can be applied to financial markets describing correlations between stock price changes [KDI00], as well as in physiology to characterize correlations between data representing the electric activity of brain [KDGO06, Šeb03]. A survey on the various applications of asymmetric random matrices was undertaken in [DKI11]. Quantum chromodynamics (QCD) provides another area of application for non-hermitian random matrix ensembles in the study of the Dirac operator spectrum with chemical potential μ. See [Ver11] for a review of the field.

QCD gave rise to the study of the chiral counterpart of the Ginibre ensemble, which was recently completely solved [APS09a, AKP10, APS10]. A summery was recently undertaken in [Ake11]. In addition the application of the one-component plasma on a two-dimensional surface led to the study of the complex spherical ensemble as well as the truncations of unitary Haar distributed matrices [FK09]. Their real counterparts were investigated in [FM11] and [KSŻ10, **?**]. A generaliza-



(a) 100 points uniformly distributed on the disk

(b) 100 eigenvalues of a complex Ginibre matrix

Figure 1.2: Point distributions in the plane

tion of the Ginibre ensemble was introduced in [FKS97a, FKS97b]. It consisted of matrices of the form $G = H_1 + i\alpha H_2$, with H_1, H_2 being two independent GUE matrices and $0 \leq \alpha \leq 1$. It's real counterpart the partly symmetric Ginibre ensemble was introduced and solved in [FN08].

Beyond matrices with known or even Gaussian matrix measure few things are known in non-hermitian random matrix theory. For matrices with independent entries Girko's circular law holds:

Theorem 1.2.2 ([TV08])**.** *Let $A_n \in \mathbb{C}^{N\times N}$ and let the empirical spectral measure (ESD) of $\frac{1}{\sqrt{N}}A_N$ be defined as:*

$$\mu_{\frac{1}{\sqrt{N}}A_N}(s,t) := \frac{1}{N}\Big|\{1 \leq i \leq N,\, \mathrm{Re}(\lambda_i) \leq s\,, \mathrm{Im}(\lambda_i) \leq t\,\}\Big|. \tag{1.2.6}$$

Now let the entries of A_N be i.i.d. with mean zero and variance 1. Then the ESD of A_N converges to the uniform measure in the unit disk almost surely in the limit of large N.

The circular law was first introduced in [Gir85] and then extended by [Bai97]. In its current form it was proved in [TV08]. For matrices with polynomial potential the single-ring theorem of Feinberg and Zee holds true.

Theorem 1.2.3 ([GZ11, FZ97])**.** *Let V denote a polynomial with positive leading coefficient. Let $X_N \in \mathbb{C}^{N\times N}$ be distributed according to the law:*

$$\frac{1}{Z_N}\,\mathrm{e}^{-N\,\mathrm{tr}\left(V(XX^\dagger)\right)}\,dX, \tag{1.2.7}$$

where Z_N is the normalization constant and dX is the Lebesgue measure on the space of $N \times N$ matrices. Let μ_{X_N} be the ESD of X_N. Then the following hold:

1. *μ_{X_n} converges in probability to a limiting probability measure μ_A.*
2. *The support of μ_A is a single ring.*
3. *The measure μ_A possesses a radially-symmetric density ρ_A, which is constant on its support.*

The single-ring theorem was first formulated in [FZ97] and then rigorously proved in [GZ11].

Beyond these results the theory of non-hermitian random matrices is still far from being as thoroughly understood as its hermitian counterpart. Not only is the question of universality of the eigenvalue correlation functions still mainly open, in addition, due to the level of technical difficulty there are only few known examples of completely solvable non-hermitian random matrix ensembles.

Remark 1.2.4. *Recently Tao and Vu [TV12] succeeded in establishing universality for the correlation kernel of non-hermitian random matrices with jointly independent exponentially decaying entries with independent real and imaginary parts.*

Remark 1.2.5. *A special class of non-hermitian random matrices are normal matrices. A matrix A is said to be normal, if it commutes with its hermitian conjugate: $[A, A^\dagger] = 0$. Normal matrices serve as another class of examples for ensembles whose asymptotic mean eigenvalue density consists of a uniform distribution on a bounded domain in the complex plane. For details see [RTW05, IT07].*

1.3 Preliminaries

In the following section we introduce some notation as well as giving a short introduction to the theory of the wedge product.

1.3.1 Notation

I_n denotes the $n \times n$ identity matrix.
δ denotes the Dirac delta function, while $\Theta(x)$ denotes the Heaviside step function:

$$\Theta(x) = \begin{cases} 1, & \text{for } x > 0 \\ \frac{1}{2}, & \text{for } x = 0 \\ 0, & \text{for } x < 0 \end{cases} . \tag{1.3.1}$$

Special functions

The gamma function: $\Gamma(a) = \int_0^\infty \mathrm{e}^{-u}\, u^{a-1} du$.
The lower incomplete gamma function: $\gamma(a,z) = \int_0^z \mathrm{e}^{-u}\, u^{a-1} du$.
The upper incomplete gamma function: $\Gamma(a,z) = \int_z^\infty \mathrm{e}^{-u}\, u^{a-1} du$.
Note that: $\Gamma(a) = \gamma(a,z) + \Gamma(a,z)$.
The beta function: $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} du$.
The incomplete beta function: $I_z(a,b) = \frac{1}{B(a,b)} \int_0^z u^{a-1}(1-u)^{b-1} du$.
The error function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt$.
The complementary error function: $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\, dt$.
Note that: $1 = \mathrm{erf}(x) + \mathrm{erfc}(x)$.

1.3.2 Jacobians and the wedge product

Many derivations in this work rely on performing a change of variables, typically using matrix decompositions. A convenient way to compute Jacobians in this context is provided through the use of the exterior product operation as introduced below. The exterior product or wedge product is a product on an exterior algebra, providing a way to formally multiply differential forms. This following chapter is heavily inspired by [Mui82] as well as [For10b] and serves as an introduction to the theory of wedge product and Jacobians.

Definition 1.3.1 ([For10b]). *Let $dz_i(j) := \delta_{i,j} dz_i$ then define:*

$$dz_1 \wedge \cdots \wedge dz_N =: \bigwedge_{j=1}^{N} dz_i = \det[\, dz_i(j)\,]_{i,j=1,\ldots,N}. \tag{1.3.2}$$

Note in particular that the wedge product (also termed exterior product) is not commutative, but skew-symmetric implying:

$$dz_1 \wedge dz_2 = -dz_2 \wedge dz_1 \tag{1.3.3}$$

$$dz_1 \wedge dz_1 = 0. \tag{1.3.4}$$

Furthermore some additional notation is needed. From now on, for the real matrix $X \in \mathbb{R}^{N\times M}$ let dX denote the matrix of differentials:

$$dX = \begin{pmatrix} dx_{11} & \dots & dx_{1M} \\ \vdots & & \vdots \\ dx_{N1} & \dots & dx_{NM} \end{pmatrix}. \tag{1.3.5}$$

It is noteworthy that, two matrices of differentials dX, dY inherit the product rule for differentiation from their matrix elements:

$$d(XY) = dX\,Y + X\,dY. \tag{1.3.6}$$

In addition let (dX) denote the wedge product of the functionally independent entries of dX. Hence for an arbitrary $X \in \mathbb{R}^{N\times M}$:

$$(dX) = \bigwedge_{j=1}^{N} \bigwedge_{k=1}^{M} dx_{jk}. \tag{1.3.7}$$

If X is a symmetric $N \times N$ matrix then (dX) denotes the wedge product of the $\frac{1}{2}N(N+1)$ distinct elements of dX:

$$(dX) = \bigwedge_{1\le i\le j\le N} dx_{jk}. \tag{1.3.8}$$

In addition if X is a diagonal matrix, then:

$$(dX) = \bigwedge_{j=1,\dots,N} dx_{jj}. \tag{1.3.9}$$

In a similar fashion for the complex matrix $Z \in \mathbb{C}^{N\times N}$ with entries $z_{jk} = x_{jk} + iy_{jk}$ let dZ denote the matrix of complex differentials $dz_{jk} = dx_{jk} + idy_{jk}$:

$$dZ = \begin{pmatrix} dz_{11} & \dots & dz_{1N} \\ \vdots & & \vdots \\ dz_{N1} & \dots & dz_{NN} \end{pmatrix}, \tag{1.3.10}$$

while (dZ) denotes the wedge product of the functionally independent entries of

dZ. Hence for $Z \in \mathbb{C}^{N\times N}$ with entries $z_{jk} = x_{jk} + iy_{jk}$ for $j, k = 1, \ldots, N$:

$$(dZ) = \bigwedge_{j=1}^{N} \bigwedge_{k=1}^{N} (dx_{jk} \wedge dy_{jk}). \tag{1.3.11}$$

Similarly for a Hermitian matrix $Z \in \mathbb{C}^{N\times N}$:

$$(dZ) = \bigwedge_{1\le j<k\le N} (dx_{jk} \wedge dy_{jk}) \bigwedge_{j=1}^{N} dx_{jj}, \tag{1.3.12}$$

as well as an skew-hermitian matrix $Z \in \mathbb{C}^{N\times N}$ with $Z^\dagger = -Z$:

$$(dZ) = \bigwedge_{1\le j<k\le N} (dx_{jk} \wedge dy_{jk}). \tag{1.3.13}$$

In addition note that the definition of the wedge product implies for a suitable function $f : \mathbb{C}^N \to \mathbb{C}^N$:

$$\int_D f(x_1, \ldots, x_N)\, dx_1 \wedge \cdots \wedge dx_N = \int_D f(x_1, \ldots, x_N)\, dx_1 \cdots dx_N, \tag{1.3.14}$$

since only the diagonal elements in the determinant in definition 1.3.1 are non-zero. The wedge product now provides a compact way of changing variables by providing direct means for explicitly determining the Jacobian of a given change of variables. In the integration formulae only the absolute value of the Jacobian is of interest. As a consequence in the following, no specific attention is paid to the ordering of the differentials and it is assumed that $(dX), (dZ)$ are positive and thus define volume measures. More generally:

Theorem 1.3.2 ([Mui82, For10b]). *(a) Let $A \in \mathbb{R}^{N\times N}$ be a fixed, non-singular matrix and let $x, y \in \mathbb{R}^{N\times 1}$ be column vectors of real variables. If $x = Ay$, then:*

$$(dx) = |\det A|(dy). \tag{1.3.15}$$

Similarly let $X, Y \in \mathbb{R}^{N\times M}$ be matrices of real variables. If $X = AY$, then:

$$(dX) = |\det A|^M (dY). \tag{1.3.16}$$

Finally let $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{M\times M}$ be fixed, non-singular matrices. If $X = AYB$, then:

$$(dX) = |\det A|^M |\det B|^N (dY). \tag{1.3.17}$$

(b) Let $A \in \mathbb{C}^{N\times N}$ be a fixed, non-singular matrix and let $z, w \in \mathbb{C}^{N\times 1}$ be column

vectors of complex variables. If $z = Aw$, then:

$$(dz) = |\det A|^2 (dw). \tag{1.3.18}$$

Similarly let $Z, W \in \mathbb{C}^{N\times M}$ be matrices of complex variables. If $Z = AW$, then:

$$(dZ) = |\det A|^{2M} (dW). \tag{1.3.19}$$

Finally let $A \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{M\times M}$ be fixed, nonsingular matrices. If $Z = AWB$, then:

$$(dZ) = |\det A|^{2M} |\det B|^{2N} (dW). \tag{1.3.20}$$

Proof. (a) From [Mui82]: It is clear that the left hand side of (1.3.15) can be written as:

$$(dx) = p\,(A)(dy), \tag{1.3.21}$$

where $p(A)$ is a polynomial in the elements of A. The following conditions hold true:

- $p\,(A)$ is linear in every row of A.
- If the order of two differentials $\mathrm{d}\,x_i, \mathrm{d}\,x_j$ is reversed then the sign of (dx) is reversed. This is equivalent to interchanging the i-th and j-th row of A.
- $p\,(I_N) = 1$.

These conditions actually form the Weierstrass definition of a determinant [McD49] and thus $p(A) = \det(A)$. Moreover for $X = [x_1 \cdots x_M]$, $Y = [y_1 \cdots y_M]$:

$$(dX) = \bigwedge_{j=1}^{M} (dx_j) = \bigwedge_{j=1}^{M} \det(A)(dy_j),$$

implying (1.3.16). In addition now set $U = AY$. Then it can be shown that $(dX) = \det(B)^N (dU)$, while we know $(dU) = \det(A)^M (dY)$, which implies (1.3.17).

(b) Set:

$$\hat{z} = \begin{pmatrix} \mathrm{Re}(z_1) & \mathrm{Im}(z_1) \\ \vdots & \vdots \\ \mathrm{Re}(z_N) & \mathrm{Im}(z_N) \end{pmatrix}, \quad \hat{w} = \begin{pmatrix} \mathrm{Re}(w_1) & \mathrm{Im}(w_1) \\ \vdots & \vdots \\ \mathrm{Re}(w_N) & \mathrm{Im}(w_N) \end{pmatrix}. \tag{1.3.22}$$

Then $(dz) = \bigwedge_{j=1}^{N} \mathrm{d}\,\mathrm{Re}(z)_j \wedge \mathrm{d}\,\mathrm{Im}(z)_j = (d\hat{z})$ as well as $(dw) = (d\hat{w})$. Thus applying part (a) of the theorem to $\hat{z} = A\hat{w}$ gives (1.3.18). Then (1.3.19) and (1.3.20) follow using the ideas from the proof of part (a).

□

In the following we will make use of the Dyson β for expressing our results, where $\beta = 1$ corresponds to real matrix entries and $\beta = 2$ to complex matrix entries. The quaternion case $\beta = 4$ is not treated here. Another useful result is given by:

Theorem 1.3.3. *[Mui82, For10b] Let $X, Y \in \mathbb{C}^{N\times N}$ be hermitian (symmetric) matrices and let $A \in \mathbb{C}^{N\times N}$ be nonsingular with either real ($\beta = 1$) or complex entries ($\beta = 2$). Set:*

$$X = AYA^{\dagger} \tag{1.3.23}$$

Then:

$$(dX) = \det(A)^{\beta(N+2-\beta)}(dY). \tag{1.3.24}$$

Proof. The proof for $\beta = 1$ can be found in [Mui82] Chapter 2, theorem 2.1.6. It is clear that:

$$(dX) = (AdYA^{\dagger}) = p\,(A)(dY), \tag{1.3.25}$$

where $p\,(A)$ is a polynomial in the elements of A. This polynomial satisfies the equation:

$$p\,(A_1A_2) = p\,(A_1)p\,(A_2) \tag{1.3.26}$$

for all A_1, A_2, which in turn implies:

$$(dX) = \det(A)^k(dY) \tag{1.3.27}$$

for some integer k. In order to determine k, let $A = \mathrm{diag}(a, 1, \dots, 1)$. Then we compute:

$$AYA^{\dagger} = \begin{pmatrix} a^2y_{11} & ay_{12} & \cdots & ay_{1N} \\ a\bar{y}_{12} & y_{22} & \cdots & y_{2N} \\ \vdots & & & \vdots \\ a\bar{y}_{1N} & \bar{y}_{2N} & \cdots & y_{NN} \end{pmatrix}. \tag{1.3.28}$$

Noting that as Y is hermitian (symmetric), its diagonal entries are real it follows, that $p(A) = \det(A)^{\beta(N+2-\beta)}$. □

In the following we adopt the important convention, that $\int_{(Z)} f(Z)(dZ)$ shall denote the integral over all functionally independent entries of the $N \times N$ matrix Z. In addition for complex z we shall write $d^2z = d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$.

1.3.3 The Haar measure

In the following we are going to construct an invariant measure on the unitary group. The unitary group is a special case of the complex Stiefel manifold:

$$V_{M,N}^{\mathbb{C}} := \{Q \in \mathbb{C}^{N\times M} : Q^\dagger Q = I_M\} \qquad \text{for} \quad N \geq M. \tag{1.3.29}$$

The construction of an invariant measure on the orthogonal group follows the same idea and is outlined in [Mui82, Jam54]. Similarly the construction of an invariant measure on the Stiefel manifold is outlined in [Jam54]. For $M = N$ the complex Stiefel manifold corresponds to the unitary group $U(N)$ of $N \times N$ complex matrices endowed with the matrix multiplication as the group action. Take an element $Q \in U(N)$, then the condition $Q^\dagger Q = I_N$ imposes N^2 functionally independent conditions on the complex elements of Q. Thus the unitary group forms a N^2-dimensional manifold in the $2N^2$ dimensional Euclidean space $\mathbb{C}^{N\times N}$.

Now let $Q \in \mathbb{C}^{N\times N}$ be an unitary matrix meaning $Q^\dagger Q = I_N$. We can differentiate this equation and obtain:

$$dQ^\dagger Q + Q^\dagger dQ = 0 \quad \Longrightarrow \quad Q^\dagger dQ = -dQ^\dagger Q = -(Q^\dagger dQ)^\dagger. \tag{1.3.30}$$

Hence $H = Q^\dagger dQ$ is skew-hermitian with complex entries q_{jk}, $j, k = 1, \ldots, N$ and its wedge product of differentials has the following form:

$$(dH) = (Q^\dagger dQ) = \bigwedge_{j=1}^{N} d\,\mathrm{Im}(h_{jj}) \bigwedge_{j<k} d\,\mathrm{Im}(h_{jk}) \wedge d\,\mathrm{Re}(h_{jk}). \tag{1.3.31}$$

It can be easily verified that this differential form has maximum degree. Furthermore take $W \in U(N)$ then $(Q^\dagger dQ)$ is invariant under left translation $Q \to WQ$, as: $Q^\dagger dQ \to (WQ)^\dagger W dQ = Q^\dagger dQ$ and thus $(Q^\dagger dQ) \to (Q^\dagger dQ)$. In addition $(Q^\dagger dQ)$ is invariant with respect to right translation $Q \to QW^\dagger$ using $Q^\dagger dQ \to (QW^\dagger)^\dagger dQW^\dagger = WQ^\dagger dQW^\dagger$, which implies $(Q^\dagger dQ) \to (WQ^\dagger dQW^\dagger) = \det(W)^{2N-2}(Q^\dagger dQ) = (Q^\dagger dQ)$. We can now define a measure on the unitary group in the following way:

$$\mu(\mathbf{U}) := \int_{\mathbf{U}} (Q^\dagger dQ), \qquad \mathbf{U} \subseteq U(N). \tag{1.3.32}$$

Due to the invariance of the differential form this measure is also invariant with respect to left and right translation. Hence $\mu(V\mathbf{U}) = \mu(\mathbf{U}V) = \mu(\mathbf{U})$ for $V \in U(N)$. This measure is called the Haar measure and it can be shown that such a measure exists on any locally compact, topological group and that it is unique up to a positive constant. To construct a probability measure out of μ, one needs to compute the volume of the unitary group.

Similarly it can be shown, that for matrices $V \in V^{\mathbb{C}}_{M,N}$ the differential form $(V^\dagger dV)$ defines an invariant measure on the Stiefel manifold:

$$\hat{\mu}(\mathbf{V}) := \int_{\mathbf{V}} (V^\dagger dV), \qquad \mathbf{V} \subseteq V^{\mathbb{C}}_{M,N}(N). \tag{1.3.33}$$

Similarly, as shown in [Mui82], it is possible to construct invariant measures on the real Stiefel manifold and thus the orthogonal group.

1.3.4 Volume of Stiefel manifolds

The volume of particular cosets of the real and complex Stiefel manifold, as well as the volume of orthogonal and unitary group are needed for defining probability measures on various cosets of the real and complex Stiefel manifold. While these results are well known, the derivation of the volume of the complex Stiefel manifold highlights the usefulness of the wedge product approach . The approach employed in [Mui82] in order to compute the volume of the real Stiefel manifold was adapted to computing the volume of the complex Stiefel manifold. The volume of the real Stiefel manifold is given by [Mui82]:

$$\mathrm{Vol}(V^{\mathbb{R}}_{M,N}) = \frac{2^M \pi^{\frac{1}{2}MN - \frac{1}{4}M(M-1)}}{\prod_{j=1}^{M} \Gamma\big(\frac{1}{2}(N-M+j)\big)}. \tag{1.3.34}$$

which implies that the volume of the orthogonal group $O(N)$ is given by:

$$\mathrm{Vol}(O(N)) = \frac{2^N \pi^{\frac{1}{4}N(N+1)}}{\prod_{j=1}^{N} \Gamma(\frac{1}{2}j)}. \tag{1.3.35}$$

The following result is needed in order to derive the volume of the unitary group:

Lemma 1.3.4. *[For10b, Mui82] Let $Z \in \mathbb{R}^{N\times M}$ for $(\beta = 1)$ and $Z \in \mathbb{C}^{N\times M}$ for $(\beta = 2)$ with $Z = HT$ and $N \geq M$ be of rank M, where $H \in \mathbb{R}^{N\times N}$ satisfies the equation $H^T H = I_M$ and $T \in R^{N\times M}$ is upper triangular with positive diagonal elements. Then:*

$$(dZ) = \prod_{j=1}^{M} t_{jj}^{\beta(N-j-1+\frac{\beta}{2})} (dT)(H^\dagger dH). \tag{1.3.36}$$

Proof. The proof for $\beta = 1$ can be found in [Mui82] Chapter 2, page 63. Proof for $\beta = 2$ can be found in [For10b], page 92. ∎

Remark 1.3.5. *The decomposition $Z = HT$ from lemma 1.3.4 is nothing but the well-known QR decomposition with different notation.*

Note that for every matrix $Z \in \mathbb{C}^{N\times N}$ the decomposition from Lemma 1.3.4 exists and is unique. A similar decomposition is given by

Lemma 1.3.6. *[For10b, Mui82] Let $Z \in \mathbb{C}^{N\times M}$ with $N \geq M$ be of rank M and set $A = Z^\dagger Z$. Both matrices have real entries for $\beta = 1$ and complex entries for $\beta = 2$. Then*

$$(dZ) = 2^{-M} \det(A)^{\frac{\beta}{2}(N-M-2+\beta)} (dA)(H^\dagger dH). \tag{1.3.37}$$

Proof. Again for $\beta = 1$ the proof can be found in [Mui82] Chapter 2, page 66. While for $\beta = 2$ it can be found in [For10b], page 93. $\square$

Now let $Z \in \mathbb{C}^{N\times M}$ be a Gaussian matrix with independent entries $z_{jk} = u_{jk} + iv_{jk}$, whose real and imaginary part are independent and identically distributed according to $u_{jk}, v_{jk} \sim N(0, \frac{1}{2})$ for $j = 1, \ldots, N$, $k = 1, \ldots, M$. The joint probability density function of the N^2 independent entries of Z has the following form:

$$P(Z) = \pi^{-NM} e^{-\sum_{j=1}^{N}\sum_{k=1}^{M}(u_{jk}^2 + v_{jk}^2)} = \pi^{-NM} e^{-\operatorname{tr}(Z^\dagger Z)}. \tag{1.3.38}$$

Being a probability density, it clearly integrates to one:

$$\int_{(Z)} e^{-\operatorname{tr}(Z^\dagger Z)} (dZ) = \pi^{NM}. \tag{1.3.39}$$

Using Lemma (1.3.4) we can write $Z = HT$ where $H \in \mathbb{C}^{N\times M}$ is such that $H^\dagger H = I_M$ and $T \in \mathbb{C}^{M\times M}$ is an upper triangular matrix with real, positive diagonal entries t_{jj}, $j = 1, \ldots, M$ and complex upper triangular entries t_{jk}. Note that:

$$\operatorname{tr}\left(Z^\dagger Z\right) = \sum_{j<k} |t_{jk}|^2 + \sum_{j=1}^{M} t_{jj}^2. \tag{1.3.40}$$

As a consequence:

$$\int_{(Z)} e^{-\operatorname{tr}(Z^\dagger Z)}(dZ) = \int_{V^{\mathbb{C}}_{M,N}} \int_{(T)} e^{-\sum_{j<k}|t_{jk}|^2} \prod_{j=1}^{M} t_{jj}^{2(N-j)+1} e^{t_{jj}^2}(dT)(H^\dagger dH)$$
$$= \prod_{j<k} \int_{\mathbb{C}} e^{-|t_{jk}|^2} d^2 t_{jk} \prod_{j=1}^{M} \int_0^\infty e^{-t_{jj}^2} t_{jj}^{2(N-j)+1} dt_{jj} \int_{V^{\mathbb{C}}_{M,N}} (H^\dagger dH).$$
$$= 2^{-M} \pi^{MN - \frac{1}{2}M(M-1)} \prod_{j=1}^{M} \Gamma(N-j+1) \int_{V^{\mathbb{C}}_{M,N}} (H^\dagger dH). \quad (1.3.41)$$

Hence:

$$\mathrm{Vol}(V^{\mathbb{C}}_{M,N}) = \frac{2^M \pi^{MN-\frac{1}{2}M(M-1)}}{\prod_{j=1}^{M} \Gamma(N-M+j)}. \quad (1.3.42)$$

As a consequence we obtain the volume of the unitary group:

$$\int_{U(N)} (Q^\dagger dQ) = \frac{2^N \pi^{\frac{1}{2}N(N+1)}}{\prod_{j=1}^{N} \Gamma(j)}. \quad (1.3.43)$$

Finally we can now define the probability measure $\tilde{\mu}$ on $U(N)$ using the differential form $(dH) = \frac{1}{\mathrm{Vol}[U(N)]}(Q^\dagger dQ)$:

$$\tilde{\mu}(\mathbf{U}) := \int_{\mathbf{U}} (dH) = \frac{1}{\mathrm{Vol}[U(N)]} \int_{\mathbf{U}} (Q^\dagger dQ), \qquad \mathbf{U} \subseteq U(N). \quad (1.3.44)$$

1.3.5 Useful matrix decompositions and their Jacobians

Using the formalism provided by the wedge product, it is possible to elegantly compute the Jacobian of several matrix decompositions, which will be needed along the way. The central theme of this work is the asymptotic properties of eigenvalues of non-hermitian random matrix ensembles. Key to the derivation of the eigenvalues distributions are matrix decompositions involving the spectrum. In general the eigenvalues of a matrix $A \in \mathbb{C}^{N\times N}$ are the N roots of the characteristic polynomial $\chi(z) = \det\big(zI_N - A\big)$. As a polynomial of degree N has exactly N solutions in $\mathbb{C}$, every matrix has N (not always distinct) eigenvalues. Complex hermitian matrices have the special property of having only real eigenvalues. The main difference between matrices with complex entries and matrices with real entries lies in structure of their spectrum. In general matrices with real entries possess real eigenvalues, as well as conjugate pairs of complex eigenvalues. Again real symmetric matrices have the property of only possessing real eigenvalues. As a result the spectral decompositions for hermitian and symmetric matrices are particularly simple. Below is an overview of the most important matrix decompositions used in this work. Furthermore the Jacobian of the change of variables induced by these decompositions are stated.

Spectral decomposition

Theorem 1.3.7. *(a) Every nonsingular symmetric matrix $A \in \mathbb{R}^{N\times N}$ can be decomposed as:*

$$A = Q\Lambda Q^T, \tag{1.3.45}$$

where Q is an orthogonal matrix whose columns correspond to the eigenvectors of A with norm one, while $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ is a diagonal matrix containing the eigenvalues of A.

(b) Every nonsingular hermitian matrix $A \in \mathbb{C}^{N\times N}$ can be decomposed as:

$$A = U\Lambda U^\dagger, \tag{1.3.46}$$

where U is a unitary matrix whose columns correspond to the eigenvectors of A with norm one, while $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ is a diagonal matrix containing the real eigenvalues of A.

The spectral decomposition is not unique. In order to make (1.3.45) unique it is necessary to order the eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$. In the following this shall be possible, as the matrices with non-distinct eigenvalues form a set of measure zero. Furthermore there is freedom in the choice of the eigenvector corresponding to each eigenvalue and thus in the choice of Q. As a result the decomposition can be made unique by choosing Q with positive first row, which is equivalent to choosing Q from the right coset $O[N] := O(N)/O_d(N)$ of the orthogonal group $O(N)$. Here $O_d(N)$ denotes the orthogonal diagonal matrices of size N. Similarly (1.3.45) is made unique by ordering the eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ of A and choosing U from the right coset $U[N] := U(N)/U_d(N)$ of the unitary group $U(N)$. Here $U_d(N)$ denotes the unitary diagonal matrices of size N. Further on we shall adopt the following important convention:

Definition 1.3.8. *Let $Q \in U[N]$, then:*

$$(Q^\dagger dQ) := \begin{cases} \wedge_{1\le k<j\le N}(\mathrm{Re}(Q^\dagger dQ)_{jk} \wedge \mathrm{Im}(Q^\dagger dQ)_{jk}) & (\beta = 2), \\ \wedge_{1\le k<j\le N}(Q^\dagger dQ)_{jk} & (\beta = 1), \end{cases} \tag{1.3.47}$$

defines a maximum degree form on the manifold $U[N]$.

Theorem 1.3.9. *With the conditions of theorem 1.3.7:*

$$(dA) = \prod_{j<k} |\lambda_j - \lambda_k|^\beta (d\Lambda)(Q^\dagger dQ), \tag{1.3.48}$$

where $(Q^\dagger dQ)$ taken as in definition 1.3.8.

Singular value decomposition

As the spectrum of a matrix is only defined for square matrices, the following singular value decomposition is extremely useful when dealing with rectangular matrices.

Lemma 1.3.10. *[Singular value decomposition]*

(a) Any complex $M \times N$ matrix A with $M \geq N$ can be decomposed as:

$$A = U\Sigma V^\dagger, \tag{1.3.49}$$

where $U \in V^{\mathbb{C}}_{N,M}$ and $V \in U(N)$ is unitary. The matrix $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$ is diagonal and the singular values are ordered $\sigma_1 \geq \ldots \geq \sigma_N \geq 0$ and positive. The columns of U are the eigenvectors of $AA^\dagger$ while the columns of V are the eigenvectors of $A^\dagger A$. In addition $\sigma_1^2, \ldots, \sigma_N^2$ are the eigenvalues of $A^\dagger A$.

(b) Any real $M \times N$ matrix A with $M \geq N$ can be decomposed as:

$$A = U\Sigma V^T, \tag{1.3.50}$$

where $U \in V^{\mathbb{R}}_{N,M}$ and $V \in O(N)$ is orthogonal. The matrix $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$ is diagonal and the singular values are ordered $\sigma_1 \geq \ldots \geq \sigma_N \geq 0$ and positive. The columns of U are the eigenvectors of AA^T while the columns of V are the eigenvectors of $A^T A$. In addition $\sigma_1^2, \ldots, \sigma_N^2$ are the eigenvalues of $A^\dagger A$.

The singular values of a matrix are distinct with probability one and thus we can write: $\sigma_1 > \ldots > \sigma_N > 0$. Again the singular value decomposition is not unique. As the columns of V are eigenvectors of $A^\dagger A$, V is only defined up to a phase factor (or sign for real matrices). To make the choice of V unique we shall impose the condition that the first non-zero entry in each column of V is positive. As a consequence the matrix $U = AV\Sigma^{-1}$ and thus the decomposition are uniquely defined.

Lemma 1.3.11. *With the conditions of theorem 1.3.10 with $\beta = 1$ denoting the real matrix entries and $\beta = 2$ denoting complex matrix entries*

$$(dA) = \prod_{i<j} (\sigma_i^2 - \sigma_j^2)^\beta (U^\dagger dU)(V^\dagger dV)(d\Sigma) \tag{1.3.51}$$

where $(U^\dagger dU)$ is taken as in definition 1.3.8 and $(V^\dagger dV)$ is a differential form of maximum degree on the respective Stiefel manifold.

Proof. We start from:

$$(dA) = \left(d\left(U\Sigma V^\dagger\right)\right) = \left(dU\Sigma V^\dagger + Ud\Sigma V^\dagger + U\Sigma dV^\dagger\right) \tag{1.3.52}$$

Using $\left(U^\dagger dAV\right) = (dA)$ for orthogonal matrices U, V as well as the skew-symmetry of $U^\dagger dU =: dH$ and $V^\dagger dV =: d\hat{H}$ leads to:

$$(dA) = \left(U^\dagger dU\Sigma + d\Sigma - \Sigma V^\dagger dV\right). \tag{1.3.53}$$

Hence we need to evaluate:

$$\begin{aligned}
&\left(U^\dagger dU\Sigma - \Sigma V^\dagger dV\right) \\
=&\Bigg(\begin{pmatrix} 0 & \sigma_2 dh_{12} & \sigma_3 dh_{13} & \cdots & \sigma_N dh_{1N} \\ -\sigma_1 dh_{12} & 0 & \sigma_3 dh_{23} & \cdots & \sigma_N dh_{2N} \\ -\sigma_1 dh_{13} & -\sigma_2 dh_{23} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \sigma_N dh_{N-1,N} \\ -\sigma_1 dh_{1N} & -\sigma_2 dh_{2N} & \cdots & -\sigma_{N-1} dh_{N-1,N} & 0 \end{pmatrix} \\
&- \begin{pmatrix} 0 & \sigma_1 d\hat{h}_{12} & \sigma_1 d\hat{h}_{13} & \cdots & \sigma_1 d\hat{h}_{1N} \\ -\sigma_2 d\hat{h}_{12} & 0 & \sigma_2 d\hat{h}_{23} & \cdots & \sigma_2 d\hat{h}_{2N} \\ -\sigma_3 d\hat{h}_{13} & -\sigma_3 d\hat{h}_{23} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \sigma_{N-1} d\hat{h}_{N-1,N} \\ -\sigma_N d\hat{h}_{1N} & -\sigma_N d\hat{h}_{2N} & \cdots & -\sigma_N d\hat{h}_{N-1,N} & 0 \end{pmatrix} \Bigg) \\
=& \prod_{i<j} \left(\sigma_j^2 - \sigma_i^2\right)^\beta \bigwedge_{i \le j} dh_{ij} \bigwedge_{i<j} d\hat{h}_{ij}
\end{aligned}$$

$\square$

The Schur decomposition

As the aim of this work is the study of complex non-hermitian as well as real asymmetric matrices the spectral decomposition in theorem 1.3.7 is not applicable. A non-hermitian analog is provided by the Schur decomposition.

Theorem 1.3.12. *[Complex Schur decomposition, [Ede97]] The Schur decomposition of a complex non-singular matrix $A \in \mathbb{C}^{N\times N}$ is a matrix decomposition of the form*

$$A = U\,(S + \Lambda)\,U^\dagger, \tag{1.3.54}$$

where $U \in \mathbb{C}^{N\times N}$ is an unitary matrix and $S \in \mathbb{C}^{N\times N}$ is a strictly upper triangular matrix, while $\Lambda \in \mathbb{C}^{N\times N}$ is a diagonal matrix containing the eigenvalues of A.

Remark 1.3.13. *Note that the complex Schur decomposition is not unique. It is possible to ensure the uniqueness of the Schur decomposition as follows. Firstly the eigenvalues of A can be ordered for example by their real parts. This is possible as the eigenvalues of a non-singular matrix A are distinct with probability one. Still the decomposition is not unique, as it is possible to multiply U with any matrix of the form $D = \mathrm{diag}(\mathrm{e}^{i\phi_1}, \ldots, \mathrm{e}^{i\phi_N})$ and leave the decomposition unchanged. Hence we fix U by choosing the first non-zero coefficient of every column vector of U to be positive. This restricts the range of U to the coset $U[N] := U(N)/U_d(N)$ of the unitary group $U(N)$.*

Lemma 1.3.14. *With the conditions of theorem 1.3.12 and remark 1.3.13 the Jacobian of the change of variables induced by the Schur decomposition is given by:*

$$(dA) = \prod_{j<k} |\lambda_j - \lambda_k|^2 (d\Lambda)(dS)(U^\dagger dU), \tag{1.3.55}$$

where $(U^\dagger dU)$ is taken as in definition 1.3.8.

Proof. The proof follows from the proof of the real Schur decomposition with $k = 0$ combined with (1.3.19). □

Theorem 1.3.15. *[Real Schur Decomposition, [Ede97]] The real Schur decomposition of a real non-singular matrix $A \in \mathbb{R}^{N\times N}$ is a matrix decomposition of the form*

$$A = QRQ^T, \tag{1.3.56}$$

where $Q \in O(N)$ is orthogonal and R is block triangular of the form:

$$R = \begin{pmatrix} \lambda_1 & \cdots & r_{1k} & r_{1,k+1} & \cdots & r_{1,N} \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \lambda_k & r_{k,k+1} & \cdots & r_{k,N} \\ 0 & \cdots & 0 & Z_1 & \cdots & r_{k+1,N} \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & Z_l \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & Z \end{pmatrix} + \begin{pmatrix} 0 & R^U \\ 0 & 0 \end{pmatrix}. \tag{1.3.57}$$

Here Λ is triangular containing the real eigenvalues $\lambda_1, \ldots, \lambda_k$ of G on its diagonal and Z is block triangular containing the 2×2 blocks:

$$Z_j = \begin{pmatrix} x_j & b_j \\ -c_j & x_j \end{pmatrix}, \quad b_j c_j > 0, \quad b_j \le c_j \quad \text{and} \quad y_j = \sqrt{b_j c_j}$$

on its block diagonal. The matrix R^U is block upper triangular.

Remark 1.3.16. *Again it is necessary to order the eigenvalues of A in order to make the decomposition unique. In addition the orthogonal matrix Q needs to be chosen from the right coset $O[N] := O(N)/O_d(N)$ of the orthogonal group $O(N)$.*

Theorem 1.3.17. *Using the conditions from theorem 1.3.15 and 1.3.16 the Jacobian of the change of variables in (1.3.56) is given by:*

$$|J| = 2^l \big|\Delta\big(\{\lambda_j\}_{j=1,\ldots,k} \cup \{x_j \pm i y_j\}_{j=1,\ldots,l}\big)\big| \prod_{i>k}(b_i - c_i) \tag{1.3.58}$$

with $\Delta\big(\{z_p\}_{p=1,\ldots,n}\big) := \prod_{i<j}(z_j - z_i)$ denoting the Vandermonde determinant.

Proof. In order to prove this statement we have to use the following result

Lemma 1.3.18. *[Ede97] Let $X \in \mathbb{C}^{M \times N}$ and let $A \in \mathbb{C}^{N \times N}$,$B \in \mathbb{C}^{M \times M}$ be square matrices with full sets of eigenvectors. We define the linear operator L:*

$$L(X) = XA - BX. \tag{1.3.59}$$

Now if λ_A is an eigenvalue of A and λ_B is an eigenvalue of B, then $\lambda_A - \lambda_B$ is an eigenvalue of the operator L.

Proof. It is possible to represent the operator L using the Kronecker product as follows: $L = A^T \otimes 1 - 1 \otimes B$. Then if v_A is a left eigenvector of A and v_B is a right eigenvector of B. Then $v_A^T v_B$ is an eigenvector of L. As A, B have full sets of eigenvectors, all eigenvectors of L are accounted for. $\square$

Now differentiating leads to:

$$dA = QdRQ^T + dQRQ^T + QRdQ^T. \tag{1.3.60}$$

In addition we set $dH = Q^T dQ$ and note that dH is anti-symmetric $(dH)^T = -dH$. As a result:

$$(dM) = (Q^T dA)Q = (dR + dHR - RdH) = (dA), \tag{1.3.61}$$

which implies:

$$(dA) = \bigwedge_{i>j}(dM_{ij}) \bigwedge_{i=j}(dM_{ii}) \bigwedge_{i>j}(dM_{ij}). \tag{1.3.62}$$

Moreover we can write:

$$\begin{aligned}
dM_{ij} &= dH_{ij}R_{jj} - R_{ii}dH_{ij} + \sum_{\nu<j} dH_{i\nu}R_{\nu j} - \sum_{\nu>j} dH_{i\nu}R_{\nu j}, && \text{for} \quad i > j \\
dM_{ij} &= dR_{jj} + \sum_{\nu<j} dH_{i\nu}R_{\nu j} - \sum_{\nu>j} dH_{i\nu}R_{\nu j}, && \text{for} \quad i = j \\
dM_{ij} &= dR_{ij} + dH_{ij}R_{jj} - R_{ii}dH_{ij} + \sum_{\nu<j} dH_{i\nu}R_{\nu j} - \sum_{\nu>j} dH_{i\nu}R_{\nu j}, && \text{for} \quad i < j.
\end{aligned}$$

Note that depending on the index the quantities in 1.3.63 are either of size $1 \times 1, 1 \times 2, 2 \times 1$ or 2×2. We start with the most difficult case $i > j$ and note that inside the summation sign the differentials $dH_{i\nu}$ and $dH_{\nu j}$ have either first index greater than i or second index smaller than j. Therefore:

$$\begin{aligned}
\bigwedge_{i>j}(dM_{ij}) = & \bigwedge_{j=N-1,i=N} \bigwedge_{j=N-2,i=N-1}^{N} \cdots \bigwedge_{j=1,i=2}^{N} \\
& \wedge \left(dH_{ij}R_{jj} - R_{ii}dH_{ij} + \sum_{\nu<j} \big(dH_{i\nu}R_{\nu j} - \sum_{\nu>j} dH_{i\nu}R_{\nu j}\big)\right. \\
= & \bigwedge_{i>j} \wedge (dH_{ij}R_{jj} - R_{ii}dH_{ij})
\end{aligned} \tag{1.3.63}$$

as each differential in the summation has already been wedged once. In addition lemma 1.3.18 gives:

$$\bigwedge_{i>j} dM_{ij} = \left|\Delta\big(\{\lambda_j\}_{j=1,\ldots,k} \cup \{x_j \pm iy_j\}_{j=1,\ldots,l}\big)\right| \bigwedge_{i>j} dH_{ij}. \tag{1.3.64}$$

Similarly for $i=j$ the differentials inside the sum vanish, yielding:

$$\bigwedge_{i=1}^{N} dM_{ii} = \bigwedge_{i=1}^{k} dM_{ii} \bigwedge_{i=k+1}^{N} dM_{ii} \tag{1.3.65}$$
$$= \bigwedge_{i=1}^{k} dM_{ii} d\lambda_i \bigwedge_{i=1}^{l} \wedge (dZ_i + dH_{k+i,k+i} Z_i - Z_i dH_{k+i,k+i})$$

Note again that dH is anti-symmetric and thus:

$$dH_{k+i,k+i} = \begin{pmatrix} 0 & dh_{k+i,k+i} \\ -dh_{k+i,k+i} & 0 \end{pmatrix}, \tag{1.3.66}$$

which gives:

$$dZ_i + dH_{k+i,k+i} Z_i - Z_i dH_{k+i,k+i} = \begin{pmatrix} dx_i + (b_i - c_i) dh_{k+i,k+i} & db_i \\ -dc_i & dx_i + (c_i - b_i) dh_{k+i,k+i} \end{pmatrix}. \tag{1.3.67}$$

As a result:

$$\bigwedge_{i=1}^{N} dM_{ii} 2^l \prod_{j=1}^{l} (b_j - c_j) dx_j db_j dc_j (d\Lambda) \bigwedge_{i=k+1}^{N} dh_{ii}. \tag{1.3.68}$$

Finally it is straightforward to show:

$$\bigwedge_{i>j} (dM_{ij}) = (dR^U). \tag{1.3.69}$$

□

Chapter 2

The inducing procedure

The aim of this work is to introduce three new classes of non-hermitian random matrix ensembles, which are then subsequently analyzed and solved. We shall start by introducing rectangular generalizations of the already known Ginibre, spherical and Jacobi ensembles. As rectangular matrices do not possess eigenvalues but only singular values, we shall introduce a method of quadratizing rectangular matrices. Rectangular matrices can be quadratized by applying a unitary transformation, which sets certain entries of the original rectangular matrix to zero. The resulting matrix consists of a rectangular matrix, which is made up of a square non-zero sub matrix and zeros. The square non-zero sub matrix is referred to as the quadratization of the rectangular matrix and it is possible to study its spectrum. Thus in this work we first introduce rectangular generalizations of the Ginibre, spherical and Jacobi ensembles, in order to apply the quadratization procedure outlined in section 2.1. These three ensembles are, besides the Chiral ensembles, the only known examples of completely solved non-hermitian random matrix ensembles.

In the limit of large matrix dimensions the eigenvalues of the Ginibre ensemble, the spherical ensemble and sub matrices of random unitary matrices are uniformly distributed, after appropriate projections: on the plane, the sphere and the pseudo-sphere, respectively. As already noted in [FK09] page 1, line 1: "The plane, sphere and pseudo-sphere are special geometries in a number of related many body statistical systems". In particular the lowest Landau level wave function for quantum particles in a magnetic field and the particle interaction of a two-dimensional one-component plasma (see section 2.4 for a more detailed description) at inverse temperature $\beta = 2$ gives rise to solvable states in these geometries [Cai81, FK09]. In addition these geometries are also found when analyzing zeros of random polynomials [FH99, Leb00, For10a].

The quadratization procedure then induces a new matrix measure on the square non-zero quadratizations of these three ensembles resulting in three new classes of non-hermitian random matrix ensembles, which are here-after referred to as the induced family of non-hermitian random matrix ensembles. The main part of this work is then concerned with solving the induced family of non-hermitian random matrix ensembles, thus providing three new solvable examples of non-hermitian random matrix ensembles.

The simplest example of a non-hermitian random matrix ensemble is the case of a random matrix with i.i.d. Gaussian entries. This is known as the Ginibre ensemble.

Definition 2.0.19. *The Ginibre ensemble is the space of $M \times N$ non-hermitian (asymmetric) matrices $A \in \mathbb{C}^{M\times N}$, whose elements are independent normal random variables. It is specified by the matrix measure:*
$\mathrm{d}\,\mu_{Ginibre,\beta} = P_{Ginibre,\beta}(A^\dagger A)(dA)$ *with:*

$$P_{Ginibre,\beta}(A^\dagger A) = \left(\frac{\beta}{2\pi}\right)^{\frac{\beta}{2}MN} \mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(A^\dagger A)}\,. \tag{2.0.1}$$

If A pertains to the Ginibre ensemble, we write $A \sim \mathrm{Gin}_\beta(M,N)$.

According to the circular law, theorem 1.2.2 the rescaled eigenvalues of square Ginibre matrices are in the limit of large matrix dimensions (to leading order) uniformly distributed on the unit disk.

It is possible to use Ginibre matrices in order to introduce a new type of non-hermitian ensemble, which is referred to as the spherical ensemble. The complex spherical ensemble was first considered in [Kri09]. The the element and eigenvalue jpdf's of complex spherical random matrices are computed in [FK09, For10b], while the real spherical ensemble is extensively treated in [FM11]. A spherical matrix for $\beta = 1, 2$ can be generated as follows:

Definition 2.0.20 ([Kri09, FK09])**.** *Let $A, B \in \mathbb{C}^{N\times N}$ be either real ($\beta = 1$) or complex ($\beta = 2$) Ginibre matrices as defined in 2.0.19. Then set:*

$$Y = A^{-1}B. \tag{2.0.2}$$

The matrix Y is called a spherical matrix.

Moreover,

Theorem 2.0.21 ([Kri09, FK09]). *Let $Y \in \mathbb{C}^{N\times N}$ be either a real ($\beta = 1$) or complex ($\beta = 2$) spherical matrix as in definition 2.0.20. The element jpdf of the random matrix Y is then given by:*

$$P(Y) = C_{spherical,\beta} \det\left(I_N + YY^\dagger\right)^{-\beta N}. \tag{2.0.3}$$

After an inverse stereographical projection the eigenvalues of Y are to leading order uniformly distributed on the sphere. It is possible to generalize the spherical ensemble to rectangular random matrices. Hence consider now the rectangular matrices $a \in \mathbb{C}^{n\times N}$, $X \in \mathbb{C}^{N\times M}$ with $n \geq N$ and $M \geq N$ either pertaining to the real ($\beta = 1$) or complex ($\beta = 2$) rectangular Ginibre ensemble as defined in 2.0.19. Setting $A = a^\dagger a$ then creates a random matrix pertaining to the well-known Wishart ensemble.

Definition 2.0.22. *[[Mui82, For10b]] Let $a \in \mathbb{C}^{n\times N}$ with $n \geq N$ be a Ginibre matrix with either real ($\beta = 1$) or complex ($\beta = 2$) entries. Set $A = a^\dagger a$. The matrix A is then referred to as a Wishart matrix with parameters n, N and we write $A \sim W_N^\beta(n)$.*

Lemma 2.0.23 ([Mui82, For10b]). *The element jpdf of a Wishart matrix $A \sim W_N^\beta(n)$ is given by:*

$$P(A) = \frac{\pi^{-\frac{\beta}{4}N(N-1)}\left(\frac{2}{\beta}\right)^{-\frac{1}{2}nN}}{\prod_{j=1}^{N}\Gamma\left(\frac{\beta}{2}(n-N+j)\right)} \det(A)^{\frac{\beta}{2}(n-N-2+\beta)}\, \mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(A)} \tag{2.0.4}$$

Proof. Write $A = Z^\dagger Z$, where $Z \sim \mathrm{Gin}_\beta(n, N)$. The density of Z is given by:

$$P(Z) = \left(\frac{2\pi}{\beta}\right)^{-\frac{\beta}{2}nN} \mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(Z^\dagger Z)}. \tag{2.0.5}$$

Now set $Z = H_1 T$ as in lemma 1.3.4, then $A = T^\dagger T$ as well as:

$$(dZ) = 2^{-N}\det(A)^{\frac{\beta}{2}(n-N-2+\beta)}(dA)(H_1^\dagger dH_1), \tag{2.0.6}$$

where $(H_1^\dagger dH_1)$ denotes a maximum degree differential form on either the real or complex Stiefel manifold. The joint density of A, H_1 is given by:

$$P(A, H_1) = 2^{-N}\left(\frac{2\pi}{\beta}\right)^{-\frac{\beta}{2}nN}\det(A)^{\frac{\beta}{2}(n-N-2+\beta)}\,\mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(A)}. \tag{2.0.7}$$

Integrating out H_1 using the volume of the respective Stiefel manifold from equations (1.3.34) and (1.3.42) yields the desired result. $\square$

Using the just introduced Wishart matrices we shall generalize the spherical ensemble to rectangular matrices. Incidently these rectangular generalizations of the spherical ensemble pertain to the so-called matrix-variate t-distribution. The matrix-variate t-distribution and its complex counterpart are actually matrix variate generalizations of the student t distribution and were already introduced in multivariate statistics by Dickey in 1967, [Dic67]. Using Wishart and Ginibre matrices they can be generated as follows:

Definition 2.0.24. *[[Dic67]] Let $A \sim W_N^\beta(n)$ pertain to the Wishart ensemble with parameters n, N and let $X \sim \mathrm{Gin}_\beta(M,N)$ pertain to the Ginibre ensemble. Furthermore let us form a random matrix $Y \in \mathbb{C}^{M\times N}$ by setting:*

$$Y = XA^{-\frac{1}{2}}. \tag{2.0.8}$$

Then Y is said to pertain to the complex rectangular spherical ensemble with parameters n, N, M and we write $Y \sim T_N^\beta(n, M)$.

Note that the term "generalization" is used rather loosely in the context of the spherical ensemble and its rectangular counterpart the matrix-variate t-distribution. Choosing the parameters $n = M = N$ in definition 2.0.24 yields a random matrix Y, whose element jpdf coincides with the element jpdf of a spherical matrix from definition 2.0.20. This is due to the relation:

$$A \sim U\sqrt{A^\dagger A} \tag{2.0.9}$$

for square Ginibre matrices A and Haar distributed matrices U. However the spherical matrices from definition 2.0.20 only coincide in probability with a square matrix from definition 2.0.24.

Theorem 2.0.25. *The element jpdf of a rectangular spherical matrix $Y \sim T_N^\beta(n, M)$ from definition 2.0.24 is given by:*

$$P_{\text{Spherical},\beta}(Y^\dagger Y) = \pi^{-\frac{\beta}{2}MN} \prod_{j=1}^{N} \frac{\Gamma\left(\frac{\beta}{2}(n+M-N+j)\right)}{\Gamma\left(\frac{\beta}{2}(n-N+j)\right)} \det\left(I_N + YY^\dagger\right)^{-\frac{\beta}{2}(n+M)}. \tag{2.0.10}$$

Proof. The joint density of A, X is given by:

$$P(A,X) = \frac{\left(\frac{2}{\beta}\right)^{-\frac{1}{2}MN-\frac{1}{2}nN}\pi^{-\frac{\beta}{2}MN-\frac{\beta}{4}N(N-1)}}{\prod_{j=1}^N \Gamma\left(\frac{\beta}{2}(n-N+j)\right)} \det(A)^{\frac{\beta}{2}(n-N-2+\beta)}\, \mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(A)}\, \mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(XX^\dagger)}.$$

Furthermore change variables $X = YA^{\frac{1}{2}}$ with Jacobian $\det(A)^{\frac{\beta}{2}M}$ yielding:

$$\begin{aligned}P(A,Y) =& \frac{(\frac{2}{\beta})^{-\frac{1}{2}MN-\frac{1}{2}nN}\pi^{-\frac{\beta}{2}MN-\frac{\beta}{4}N(N-1)}}{\prod_{j=1}^{N}\Gamma\big(\frac{\beta}{2}(n-N+j)\big)} \times \\ & \det(A)^{\frac{\beta}{2}(n+M-N-2+\beta)}\,\mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}\left(A^{\frac{1}{2}}(I_N+YY^{\dagger})A^{\frac{1}{2}}\right)}. \end{aligned} \tag{2.0.11}$$

Another change of variables $H = A^{\frac{1}{2}}(I_N + YY^{\dagger})(A^{\frac{1}{2}})^{\dagger}$ with Jacobian $\det(I_N + YY^{\dagger})^{\frac{\beta}{2}(N+2-\beta)}$ gives:

$$P(A,Y) = \frac{(\frac{2}{\beta})^{-\frac{1}{2}MN-\frac{1}{2}nN}\pi^{-\frac{\beta}{2}MN-\frac{\beta}{4}N(N-1)}}{\prod_{j=1}^{N}\Gamma\big(\frac{\beta}{2}(n-N+j)\big)}\frac{\det(H)^{\frac{\beta}{2}(n+M-N-2+\beta)}}{\det\left(I_N+YY^{\dagger}\right)^{\frac{\beta}{2}(n+M)}}\,\mathrm{e}^{-\frac{\beta}{2}\operatorname{tr}(H)}.$$

Then integrating out H using the normalization for Wishart matrices with parameters $n + M, N$ gives the desired result. □

Finally it remains to introduce the Jacobi ensemble. For this purpose consider a random matrix $Q \in \mathbb{C}^{K\times K}$ pertaining to the Circular Unitary ensemble, meaning that, Q is unitary: $Q^{\dagger}Q = I$ and distributed according to the Haar measure on the unitary group.

Definition 2.0.26. *The unitary group $U(N)$ equipped with the Haar measure $\mathrm{d}\,\mu$ forms the Circular Unitary ensemble (CUE).*

The eigenvalues of a unitary matrix lie on the unit circle: $\lambda_1 = \mathrm{e}^{i\phi_1}, \ldots, \lambda_N = \mathrm{e}^{i\phi_N}$. Furthermore,

Theorem 2.0.27 ([Meh04, For10b])**.** *The eigenvalue jpdf of a complex $N \times N$ matrix pertaining to the CUE is given by:*

$$p_{CUE}(\lambda_1,\ldots,\lambda_N) = \frac{1}{N!\big(2\pi\big)^{N}}\prod_{1\le j<k\le N}\big|\,\mathrm{e}^{i\phi_k}-\mathrm{e}^{i\phi_j}\,\big|^2. \tag{2.0.12}$$

Now consider sub matrices of these random unitary matrices with either real or complex entries. The motivation in studying truncations of random unitary matrices stems from chaotic scattering problems, where truncations of unitary matrices model the transmission matrices [Bee97]. Moreover truncations of random orthogonal matrices are found in applications such as describing the quasiparticle excitations in metals and superconductors [AZ97, DBB10] and in the context of quantum maps performed on real quantum states [BŻ06]. Square truncations of CUE matrices were first studied in [SŻ00], while the element jpdf of rectangular truncations was given in [For06]. Square truncations of random orthogonal matrices were first studied in [KSŻ10]. Consider now the rectangular matrix $A_{M\times N}\in\mathbb{C}^{M\times N}$ obtained by eliminating $l_M=K-M$ rows and $l_N=K-N$ columns from the matrix:

$$Q=\begin{pmatrix} A_{M\times N} & B_{M\times l_N}\\ C_{l_M\times N} & D_{l_M\times l_N}\end{pmatrix}. \tag{2.0.13}$$

Note that due to the condition $A^\dagger_{M\times N}A_{M\times N}\leq 1$ all singular values of $A_{M\times N}$ lie inside the unit disk. Then,

Theorem 2.0.28. *Let $Q\in\mathbb{C}^{K\times K}$ from equation (2.0.13) be a random unitary matrix picked uniformly at random from either the unitary group ($\beta=2$) or the orthogonal group ($\beta=1$). The joint element pdf of the sub matrices $A_{M\times N}, C_{l_M\times N}$ with $M\geq N$ is then given by:*

$$P_{\mathit{Stief},\beta}\left(A_{M\times N},C_{l_M\times N}\right)=c_{\mathit{Stief},\beta}\,\delta\big(A^\dagger_{M\times N}A_{M\times N}+C^\dagger_{l_M\times N}C_{l_M\times N}-I_N\big) \tag{2.0.14}$$

with normalization constant:

$$c_{\mathit{Stief},\beta}=\frac{\prod_{j=1}^{N}\Gamma\big(\frac{\beta}{2}(K-N+j)\big)}{2^N\pi^{\frac{\beta}{2}KN-\frac{\beta}{4}N(N-1)}}. \tag{2.0.15}$$

Proof. As Q is unitary (orthogonal) the following condition for $A_{M\times N}, C_{l_M\times N}$ holds:

$$A^\dagger_{M\times N}A_{M\times N}+C^\dagger_{l_M\times N}C_{l_M\times N}=I_N. \tag{2.0.16}$$

In addition note that the matrix:

$$H=\begin{pmatrix} A_{M\times N}\\ C_{l_M\times N}\end{pmatrix} \tag{2.0.17}$$

consists of the first N columns of the unitary matrix Q and thus is an element of either the complex or real Stiefel manifold. The volume of the Stiefel manifolds is given in (1.3.34) and (1.3.42). □

Integrating out the matrix $C_{l_M\times N}$ gives the element jpdf of the rectangular truncations $A_{M\times N}$ for $K>M+N$:

Theorem 2.0.29. *[[BM94, Bee97, JLPB94, For06, SŻ00]] The element jpdf of the matrix $A\in\mathbb{R}^{M\times N}$ obtained by eliminating $l_M=K-M$ rows and $l_N=K-N$ columns with $K\geq M+N$ from the matrix $Q\in U(N)$ for ($\beta=2$) or $Q\in O(N)$ for ($\beta=1$) is given by:*

$$P_{\mathit{Jacobi},\beta}(A^\dagger A)=\gamma_{\mathit{Jacobi},\beta}\det\left(I_N-A^\dagger A\right)^{\frac{\beta}{2}(K-M-N+1-\frac{2}{\beta})}, \tag{2.0.18}$$

where

$$\gamma_{Jacobi,\beta} = \pi^{-MN} \prod_{j=1}^{N} \frac{\Gamma\big(\frac{\beta}{2}(K-N+j)\big)}{\Gamma\big(\frac{\beta}{2}(K-M-N+j)\big)}. \tag{2.0.19}$$

Proof. The first step involves rewriting the matrix δ function using the Fourier representation of the individual δ functions. Then:

$$I = \int_{(C)} \delta\big(A^\dagger A + C^\dagger C - I_N\big)(dC) \propto \int_{(C)} \int_{(H)} e^{i\,\mathrm{tr}\,\left(H(A^\dagger A + C^\dagger C - I_N)\right)}(dH)(dC) \tag{2.0.20}$$

where H is a $N \times N$ hermitian (symmetric) with diagonal elements h_{jj} and off-diagonal elements $\frac{1}{2}h_{ij}$. Our aim is to apply [For10b], Proposition 3.2.8, which makes it necessary to introduce the perturbation:

$$H = \lim_{\mu \to 0^+} \big(H - \mu I_N\big). \tag{2.0.21}$$

As a consequence:

$$I \propto \lim_{\mu \to 0^+} \int_{(C)} \int_{(H)} \mathrm{e}^{i\,\mathrm{tr}\,\left((H-\mu I_N)(A^\dagger A + C^\dagger C - I_N)\right)}(dH)(dC). \tag{2.0.22}$$

Furthermore change variables $W = C^\dagger C$ with Jacobian $|J| \propto \det(W)^{\frac{\beta}{2}(l_M - N - 2 + \beta)}$, then:

$$I \propto \lim_{\mu \to 0^+} \int_{(W)} \int_{(H)} \det(W)^{\frac{\beta}{2}(l_M - N - 2 + \beta)}\, \mathrm{e}^{i\,\mathrm{tr}\,\left((H-\mu I_N)(A^\dagger A + W - I_N)\right)}(dH)(dW). \tag{2.0.23}$$

An additional change of variables $J = (H - \mu I_N)W$ with Jacobian $\det\big(H - \mu I_N\big)^{\frac{\beta}{2}l_N}$ yields:

$$I \propto \lim_{\mu \to 0^+} \int_{(J)} \int_{(H)} \mathrm{e}^{i\,\mathrm{tr}(J)} \det(J)^{\frac{\beta}{2}(l_M - N - 2 + \beta)} \times \tag{2.0.24}$$

$$\det\big(H - \mu I_N\big)^{\frac{\beta}{2}l_N} e^{i\,\mathrm{tr}\,\left((H-\mu I_N)(A^\dagger A - I_N)\right)}(dH)(dJ). \tag{2.0.25}$$

The integral over J just gives a constant which relates to the normalization of the real Wishart ensemble and hence:

$$I \propto \lim_{\mu \to 0^+} \int_{(H)} \det\big(H - \mu I_N\big)^{\frac{\beta}{2}l_N} e^{i\,\mathrm{tr}\,\left((H-\mu I_N)(A^\dagger A - I_N)\right)}(dH), \tag{2.0.26}$$

is in the right form in order to apply [For10b], Proposition 3.2.8. In addition the normalization constant can now be computed through:

$$\left(\gamma_{\text{Jacobi},\beta}\right)^{-1} = \int_{(A)} \det\left(I_N - A^\dagger A\right)^{\frac{\beta}{2}(K-M-N+1-\frac{2}{\beta})}(dA). \tag{2.0.27}$$

Start by changing variables $AA^\dagger = W$ with Jacobian:

$$|J| = 2^{-N}\,\mathrm{Vol}\left(U(N)\right) = \frac{\pi^{\frac{1}{2}N(N+1)}}{\prod_{j=1}^{N}\Gamma(j)}. \tag{2.0.28}$$

Then change variables to the eigenvalues of W. The resulting integral can then be solved by using the Selberg integral formula, theorem D.1.1. □

For $K \le M+N$ the element jpdf of A is singular (contains δ functions) due to finite mass of the boundary of the matrix ball $A^\dagger A$. The eigenvalues of a square truncation of a random unitary matrix are in the limit of large matrix dimension to leading order uniformly distributed on the pseudo-sphere [FK09]. The pseudo-sphere is a two-dimensional hyperbolic space with negative Gaussian curvature. It is defined as the upper branch of the equation:

$$-x^2 + y^2 + z^2 = -R^2. \tag{2.0.29}$$

Equipped with these three ensembles of rectangular random matrices we proceed to define our quadratization procedure.

2.1 Quadratization of rectangular matrices [FBK+11]

Consider a rectangular matrix $X \in \mathbb{C}^{M\times N}$ with M rows and N columns, where $M > N$. Thus X is a "tall" matrix and it is possible to write X in the following block matrix form:

$$X = \begin{bmatrix} Y \\ Z \end{bmatrix} \tag{2.1.1}$$

with Y denoting the $N\times N$ upper rectangular part of X, while Z denotes the $(M-N)\times N$ lower rectangular part of Z. Since the standard definition of the spectrum does not work for non–square matrices we provide a unitary transformation $W \in U(M)$ intended to set the lower block Z to zero :

$$W^\dagger X = W^\dagger \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}. \tag{2.1.2}$$

The square matrix $G \in \mathbb{C}^{N\times N}$ is referred to as the quadratization of X. Taking the unitary matrix W from the coset $U(M)/(U(M)\times U(M-N))$ of the unitary group $U(N)$ the decomposition (2.1.2) becomes unique for "tall" rectangular matrices X of full rank. One can easily find such transformations. Assuming that the matrix X has rank N, consider the linear span $\mathcal{S}$ of the column-vectors of X. Let $q_1,\dots q_N$ be an orthonormal basis in $\mathcal{S}$, and $q_{N+1},\dots q_M$ be an orthonormal basis in $\mathcal{S}^\perp$, the orthogonal complement of $\mathcal{S}$ in $\mathbb{C}^M$. If we set $W=[q_1\dots q_M]$ then (2.1.2) holds. Obviously, all other suitable unitary transformations are obtained from this W by multiplying it to the right by the block diagonal unitary matrices $\mathrm{diag}[U,V]$ where U and V run through the unitary groups $U(N)$ and $U(M-N)$, respectively. Multiplying W by $\mathrm{diag}[I_N,V]$ corresponds to choosing a different orthonormal basis in $\mathcal{S}^\perp$, and multiplying W by $\mathrm{diag}[U,I_{M-N}]$ corresponds to replacing matrix G by UG.

It is straightforward to check that any unitary matrix $W\in U(M)$ can be transformed to the block form:

$$W=\begin{bmatrix} (I_N-CC^\dagger)^{1/2} & C \\ -C^\dagger & (I_{M-N}-C^\dagger C)^{1/2}\end{bmatrix}, \quad \text{where } C \text{ is } N\times(M-N)\ , \tag{2.1.3}$$

by multiplying it to the right by block diagonal unitary matrix as above. Having settled on the choice of W, we can solve the equation in (2.1.2) for G and C.

Lemma 2.1.1. *Let $M>N$. Suppose that Y is $N\times N$ and Z is $(M-N)\times N$, and Y is invertible. Then there is a unique $M\times M$ unitary matrix W of the form (2.1.3) such that equation (2.1.2) holds. The square matrix G in equation (2.1.2) is given by:*

$$G=\left(I_N+\frac{1}{Y^\dagger}Z^\dagger Z\frac{1}{Y}\right)^{1/2} Y\ . \tag{2.1.4}$$

Proof. Assuming W as in (2.1.3), by multiplying through in (2.1.2), one obtains an equation for C. Hence:

$$Z=-\left(I_{M-N}-C^\dagger C\right)^{-1/2}C^\dagger Y=-C^\dagger\left(I_N-CC^\dagger\right)^{-1/2}Y\ . \tag{2.1.5}$$

Consequently, by making use of (2.1.2) again:

$$G=\left(I_N-CC^\dagger\right)^{1/2}Y-CZ=\left(I_N-CC^\dagger\right)^{-1/2}Y. \tag{2.1.6}$$

It is easy to check that:

$$Z^\dagger Z = Y^\dagger \left(I_N - CC^\dagger\right)^{-1} Y - Y^\dagger Y, \tag{2.1.7}$$

which implies that:

$$Y^\dagger Y + Z^\dagger Z = Y^\dagger \left(I_N - CC^\dagger\right)^{-1} Y. \tag{2.1.8}$$

This in turn allows us to write:

$$\left(I_N - CC^\dagger\right)^{-1} = \frac{1}{Y^\dagger} \left(Y^\dagger Y + Z^\dagger Z\right) \frac{1}{Y}, \tag{2.1.9}$$

which when substituted into (2.1.6) yields: (2.1.4). Note that the desired result (2.1.4) can be also rewritten in a more symmetric form:

$$G = Y \left(Y^\dagger Y\right)^{-1/2} \left(I_N + \left(Y^\dagger Y\right)^{-1/2} Z^\dagger Z \left(Y^\dagger Y\right)^{-1/2}\right)^{1/2} \left(Y^\dagger Y\right)^{1/2}, \tag{2.1.10}$$

which shows that all matrix square roots operate correctly on positive definite objects. □

We now have a procedure for quadratizing 'standing' complex rectangular matrices. Of course, in the opposite case of 'lying' rectangular matrices ($M < N$) one may apply the same procedure to quadratize the transposed matrix $X^\dagger$. Thus, any rectangular matrix X can be quadratized by a unitary transformation on its columns (or rows, if the number of columns is greater than the number of rows), giving rise to a square matrix G. As G is a unique solution of equation (2.1.2), its spectrum characterizes algebraic properties of the rectangular matrix X. Furthermore the above procedure can be repeated for real rectangular matrices $X \in \mathbb{R}^{M\times N}$. Using the same arguments as above it can be shown that any real matrix X of rank N can be uniquely quadratized using an orthogonal matrix W of the same form as in equation 2.1.3, yielding the real quadratization G of the same form as in equation (2.1.4) from lemma 2.1.1.

2.2 The induced family of real and complex random matrices

The idea of this work is to explore the concept of quadratizing real and complex rectangular matrices in the framework of random matrices. Thus in this section we specify the matrix measure induced by the quadratization procedure described in the section above.

Theorem 2.2.1. *Let $X \in \mathbb{C}^{M\times N}$ (for $\beta = 2$) and $X \in \mathbb{R}^{M\times N}$ (for $\beta = 1$) with $M \geq N$ pertain to one of the three random matrix ensembles specified by the matrix measure:*

$$\mathrm{d}\,\mu_{I,\beta}(X) = P_{I,\beta}(X^\dagger X)(dX), \tag{2.2.1}$$

where $I \in \{$Ginibre, Jacobi, Spherical$\}$ and the matrix measures are taken from in definition 2.0.19 and theorems 2.0.25, 2.0.29. Furthermore let:

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix} = U\Sigma^{\frac{1}{2}}P^\dagger \tag{2.2.2}$$

and let G be defined as in equation (2.1.4), lemma 2.1.1. Then the quadratization G of X is specified by the matrix measure:

$$\mathrm{d}\,\mu_{I,\beta}^{\mathit{Induced}}(G) = P_{I,\beta}^{\mathit{Induced}}(G)(dG), \tag{2.2.3}$$

where

$$P_{I,\beta}^{\mathit{Induced}}(G) = c_{I,\beta}^{\mathit{Induced}} \det(G^\dagger G)^{\frac{\beta}{2}(M-N)} P_{I,\beta}(G^\dagger G). \tag{2.2.4}$$

Proof. The proof is based on the singular value decomposition from lemma 1.3.10. Ignoring a set of zero probability measure, the $N\times N$ matrix $X^\dagger X$ has N distinct eigenvalues s_j, $0 < s_1 < s_2 < \ldots < s_N$, and the singular value decomposition asserts that X can be factorized as follows:

$$X = Q\,\Sigma^{1/2}P^\dagger, \tag{2.2.5}$$

where $\Sigma = \mathrm{diag}(s_1, \ldots, s_N)$, and Q and P are, respectively, $M \times N$ and $N \times N$ matrices with orthonormal columns, so that $Q^\dagger Q = P^\dagger P = I_N$. The columns of P are in fact eigenvectors of $X^\dagger X$ and, hence, are defined up to phase factor (or sign for real matrices). To make the choice of P unique, we shall impose the condition that the first non-zero entry in each column of P is positive. Consequently, the matrix Q is also defined uniquely via $Q = XP\Sigma^{-1/2}$. In addition the matrices Q, P and Σ are mutually independent. Thus we can now explicitly compute the matrix measure induced by the quadratization of the matrix X. First note:

$$X^\dagger X = P\Sigma P^\dagger \tag{2.2.6}$$

and thus using lemma 1.3.6 implies (as the Haar measure is invariant with respect to left or right translation):

$$(d\Sigma)(P^\dagger dP)(Q^\dagger dQ) = 2^{-N}\det(X^\dagger X)^{\frac{\beta}{2}(N-M-2+\beta)}(dX), \tag{2.2.7}$$

where $(P^\dagger dP)$ is the maximum degree form from definition 1.3.8 and $(Q^\dagger dQ)$ denotes the maximum degree form on the respective Stiefel manifold. Let us introduce an additional unitary matrix U of size $N \times N$ and rewrite (2.2.5) in the form:

$$X = QUU^\dagger \Sigma^{1/2} P^\dagger = QUG. \tag{2.2.8}$$

The matrix $G = U^\dagger \Sigma^{1/2} P^\dagger$ is $N \times N$. Now choose U to be Haar unitary and independent of Q, P and Σ. Note:

$$G^\dagger G = P\Sigma P^\dagger \tag{2.2.9}$$

and again lemma 1.3.6 implies:

$$(d\Sigma)(P^\dagger dP)(Q^\dagger dQ) = 2^{-N}\det(G^\dagger G)^{\frac{\beta}{2}(N-N-2+\beta)}(dG). \tag{2.2.10}$$

Combining the two relations then yields:

$$(dX) = \det(G^\dagger G)^{\frac{\beta}{2}(M-N)}(dG)(Q^\dagger dQ). \tag{2.2.11}$$

Furthermore changing variables in the probability densities P_I from X to G, U and integrating out U then gives the induced probability density:

$$P^{\text{Induced}}_{I,\beta}(G) = c^{\text{Induced}}_{I,\beta}\det(G^\dagger G)^{\frac{\beta}{2}(M-N)}P_{I,\beta}(G^\dagger G). \tag{2.2.12}$$

□

Because of the invariance of the Haar distribution, the unitary matrix U in (2.2.8) can be absorbed into Q . In other words, we have decomposed the rectangular matrix X into the product:

$$X = \tilde{Q}G \tag{2.2.13}$$

of two independent random matrices: the rectangular matrix $\tilde{Q} := QU$ with orthonormal columns, which has uniform distribution, and a square matrix G, whose distribution is induced by the distribution of X. Decomposition (2.2.13) can also be written as:

$$X = W\begin{bmatrix} G \\ 0 \end{bmatrix}, \tag{2.2.14}$$

where W is an $M \times M$ unitary (real orthogonal for $\beta = 1$) matrix obtained from the $M \times N$ matrix $\tilde{Q}$ by appending suitable column-vectors. This is nothing else but equation (2.1.2). One can transform the matrix W to the block form of (2.1.3), so that (2.2.14) becomes:

$$X = \begin{bmatrix} (I_N - CC^\dagger)^{1/2} & C \\ -C^\dagger & (I_{M-N} - C^\dagger C)^{1/2} \end{bmatrix} \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}, \tag{2.2.15}$$

where $\tilde{G} = \tilde{U} G$ for some unitary $\tilde{U}$. Obviously, $\tilde{G}$ has the same distribution as G. Thus:

Theorem 2.2.2. *Let $X \in \mathbb{C}^{M \times N}$ for $\beta = 2$ or $X \in \mathbb{R}^{M \times N}$ for $\beta = 1$ be specified as in theorem 2.2.1. Then its quadratization G is specified by the matrix measure $\mathrm{d}\,\mu_{I,\beta}^{induced}$ in equation (2.2.3), theorem 2.2.1.*

Theorem 2.2.1 together with lemma 2.1.1 provide a recipe for generating induced random matrices starting with matrices pertaining to the original distribution P_I. Interestingly, by rearranging equation (2.2.8) one obtains another recipe, which might be less efficient computationally, but is still interesting from a theoretical point of view. Indeed, since $(X^\dagger X)^{1/2} = P \Sigma^{1/2} P^\dagger$, it follows from equation (2.2.8) that:

$$G = U^\dagger Q^\dagger X = U^\dagger P^\dagger (X^\dagger X)^{1/2} = \tilde{U}^\dagger (X^\dagger X)^{1/2} , \quad \text{where } \tilde{U} = PU . \tag{2.2.16}$$

Recalling that the Haar measure is invariant with respect to right (and left) multiplication, one arrives at the following recipe for generating matrices from the induced Ginibre distribution.

Lemma 2.2.3. *Suppose that U is $N \times N$ Haar unitary and X is $M \times N$ with $M \geq N$ pertains to a random matrix ensemble specified by the matrix measure:*

$$\mathrm{d}\,\mu_{I,\beta}(X) = P_{I,\beta}(X^\dagger X)(dX), \tag{2.2.17}$$

where $I \in \{$Ginibre, Jacobi, Spherical$\}$ and is independent of U. Then the $N \times N$ matrix $G = U(X^\dagger X)^{1/2}$ pertains to the random matrix ensemble specified by the matrix measure $\mathrm{d}\,\mu_{I,\beta}^{Induced}$.

Obviously, our arguments extend to random rectangular matrices with invariant distributions other than the three explicitly mentioned, e.g. the Feinberg-Zee distribution with density:

$$P_{\mathrm{FZ}}(X) \propto \mathrm{e}^{-\operatorname{tr} V(X^\dagger X)} , \tag{2.2.18}$$

where X is $M \times N$, $M > N$. On applying the procedure of quadratization to such an ensemble, one obtains the induced Feinberg-Zee distribution:

$$P_{\mathrm{IndFZ}}(G) \propto (\det G^\dagger G)^{\frac{\beta}{2}(M-N)} \exp[-\operatorname{tr} V(G^\dagger G)] \ . \tag{2.2.19}$$

In this work we shall concentrate on solving the induced family of non-hermitian random matrix ensembles consisting of the induced Ginibre ensemble, the induced spherical ensemble and the induced Jacobi ensemble.

Chapter 3

Complex induced non-hermitian random matrix ensembles

3.1 The complex induced Ginibre ensemble

The simplest example of an induced non-hermitian random matrix ensemble is provided by applying the inducing procedure described in chapter 2 to a rectangular $M \times N$ complex Ginibre matrix. From theorem 2.2.1:

Definition 3.1.1. *The complex induced Ginibre ensemble is specified by the matrix measure* $\mathrm{d}\,\mu^{Induced}_{Ginibre,2} = P^{Induced}_{Ginibre,2}(G)(dG)$ *with:*

$$P^{Induced}_{Ginibre,2}(G) = C^{IndGin,2}_{L} \det(GG^{\dagger})^{L}\, \mathrm{e}^{-\operatorname{tr}(GG^{\dagger})}\,, \quad L = M - N \geq 0. \tag{3.1.1}$$

Clearly setting the parameter $L = 0$ leads back to the complex Ginibre ensemble. Indeed the parameter $L = M - N$ is a measure of the mismatch of dimensions in the rectangular Ginibre matrix, used to generate the induced Ginibre ensemble. Hence in the following L is referred to as the rectangularity parameter. Even though in this context L is an integer non-negative variable, the subsequent analysis extends almost verbatim to real non-negative values of L. However as the parameter $L = M - N$ is introduced through the decomposition (2.1.2) of a rectangular matrix of dimension $M \times N$, no matrix interpretation is known for non-integer values of L. In addition note that for square matrices G and Haar distributed U:

$$G \sim U\sqrt{G^{\dagger}G}. \tag{3.1.2}$$

Lemma 3.1.2. *The element jpdf of a complex induced Ginibre matrix is correctly* *normalized using:*

$$C_L^{IndGin,2} = \frac{1}{\pi^{N^2}} \prod_{j=1}^{N} \frac{\Gamma(j)}{\Gamma(j+L)}. \tag{3.1.3}$$

Proof. The normalization constant is determined by using the singular value decomposition of $A = U\Sigma V^\dagger$ from lemma 1.3.10 in order to perform integration over the element joint probability density function.

$$\begin{aligned}\left(C_L^{\text{IndGin},2}\right)^{-1} &= \int_{(G)} \det(GG^\dagger)^L \, \mathrm{e}^{-\operatorname{tr}(GG^\dagger)} (dG) \qquad (3.1.4)\\ &= \int_{(\Sigma)} \int_{U(N)} \int_{U[N]} \prod_{j<k} (\sigma_j^2 - \sigma_k^2)^2 \prod_{j=1}^{N} \sigma_j^{2L} \, \mathrm{e}^{-\sigma_j^2} (d\Sigma)(U^\dagger dU)(V^\dagger dV)\end{aligned}$$

The integral can be further simplified using the already proven results for the volume of the unitary group and applying a simple change of variables:

$$\begin{aligned}&\left(C_L^{\text{IndGin},2}\right)^{-1}\\ =&\frac{\pi^{N^2}}{N! \prod_{j=1}^{N} \Gamma^2(j)} \int_0^\infty \dots \int_0^\infty \prod_{j<k} (s_j - s_k)^2 \prod_{j=1}^{N} s_j^{L-\frac{1}{2}} s_j^2 ds_1 \dots ds_N.\end{aligned} \tag{3.1.5}$$

The factor $\frac{1}{N!}$ is introduced by removing the ordering of the singular values. Finally the remaining integral is a consequence of the Selberg integral, which can then be evaluated using theorem D.1.3 and corollary **??**. □

In this section we shall derive the eigenvalue joint probability density function of the complex induced Ginibre ensemble. Then using the method of orthogonal polynomials we compute the general n-point correlation functions. Finally in the limit of large matrix dimensions two asymptotic regimes are analyzed: strong rectangularity and almost square matrices. In both regimes in the bulk and at the edge of the support of the eigenvalue density the correlation kernels of the complex Ginibre ensemble are recovered. A main result of this work is that in the regime of almost square matrices a new limiting correlation kernel emerges at the origin.

3.1.1 The joint eigenvalue probability density function

The eigenvalue joint probability density (jpdf) of the complex induced Ginibre ensemble is obtained from that in the complex Ginibre ensemble [Gin65, Meh04] by multiplying through by $\det(GG^\dagger) = \prod_{j=1}^{N} |\lambda_j|^{2L}$ and re-evaluating the normalization constant. More precisely starting from the element joint pdf from equation (4.1.1) we first change variables from the functionally independent entries of G to the eigenvalues of G and some auxiliary variables. The latter are then integrated out. As a result:

Theorem 3.1.3. *Let $G \in \mathbb{C}^{N\times N}$ be a random matrix pertaining to the complex induced Ginibre ensemble. Then its eigenvalue jpdf is given by:*

$$p_{IndGin,2}(\lambda_1,\ldots,\lambda_N) = c_L^{IndGin} \prod_{j<k} |\lambda_k - \lambda_j|^2 \prod_{j=1}^{N} |\lambda_j|^{2L}\, \mathrm{e}^{-|\lambda_j|^2}\,. \tag{3.1.6}$$

Remark 3.1.4. *Note that the element jpdf of the complex induced Ginibre ensemble is normalized by the constant $C_L^{IndGin,2}$, while the eigenvalue jpdf of the complex induced Ginibre ensemble is normalized by the constant c_L^{IndGin}.*

Proof. The change of variables is performed using the Schur decomposition from lemma 1.3.12: $G = U(\Lambda + S)U^\dagger$, where U is an unitary matrix, S is a strictly upper-triangular matrix and $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ contains the eigenvalues of G. The Jacobian $|J|$ of the change of variables is computed in lemma 1.3.5 yielding: $|J| = \prod_{j<k} |\lambda_k - \lambda_j|^2$. As outlined in remark 1.3.13 the change of variables becomes unique, if the eigenvalues of G are ordered by their real part and the unitary matrix U is chosen from the coset $[U](N) := U(N)/U_d(N)$. All in all we obtain:

$$\begin{aligned}
&p_{\mathrm{IndGin},2}\left(\lambda_1,\ldots,\lambda_N\right)\\
=&C_L^{\mathrm{IndGin},2} \int_{(S)} \int_{[U](N)} |J| P(URU^\dagger)(U^\dagger dU)(dS)\\
=&C_L^{\mathrm{IndGin},2} \int_{(S)} \int_{[U](N)} \prod_{j<k} |\lambda_k - \lambda_j|^2 \det\left(RR^\dagger\right)^L \mathrm{e}^{-\operatorname{tr}(RR^\dagger)}(U^\dagger dU)(dS),
\end{aligned} \tag{3.1.7}$$

where $(U^\dagger dU)$ is taken from definition 1.3.8. We note that:

$$\det\left(URU^\dagger (URU^\dagger)^\dagger\right) = \det(R)\det(R^\dagger) = \prod_{j=1}^{N} \lambda_j \prod_{j=1}^{N} \bar{\lambda}_j = \prod_{j=1}^{N} |\lambda_j|^2 \tag{3.1.8}$$

$$\operatorname{tr}\left(URU^\dagger (URU^\dagger)^\dagger\right) = \operatorname{tr}(RR^\dagger) = \sum_{j=1}^{N} |\lambda_j|^2 + \sum_{j<k} |s_{jk}|^2. \tag{3.1.9}$$

As a result the eigenvalue jpdf of the complex induced Ginibre ensemble can be written as:

$$
\begin{aligned}
&p_{\text{IndGin},2}(\lambda_1,\ldots,\lambda_N) && (3.1.10)\\
=&C_L^{\text{IndGin},2}\prod_{j<k}|\lambda_k-\lambda_j|^2\prod_{j=1}^{N}|\lambda_j|^{2L}\,\mathrm{e}^{-|\lambda_j|^2}\int_{[U](N)}(dU)\int_{(S)}e^{-\sum_{j<k}|s_{jk}|^2}(dS).
\end{aligned}
$$

The unitary group is an analytical manifold and one can think of it as the direct product of the $N(N-1)$ dimensional manifold $[U](N)$ with $T_N=\{\phi\in\mathbb{R}^N\ 0\leq\phi_1\leq 2\pi\}$. This implies the following relation between the volume of $[U](N)$ and the volume of the unitary group $U(N)$:

$$
\mathrm{Vol}([U](N))=\frac{1}{(2\pi)^N}\,\mathrm{Vol}(U(N))=\frac{\pi^{\frac{1}{2}N(N-1)}}{\prod_{j=1}^{N}\Gamma(j)}. \tag{3.1.11}
$$

In addition to that:

$$
\int_{(S)}\mathrm{e}^{-\sum_{j<k}|s_{jk}|^2}(dS)=\left(\int_{\mathbb{C}}\mathrm{e}^{-|s_{12}|^2}\,d^2s_{12}\right)^{\frac{1}{2}N(N-1)}=\pi^{\frac{1}{2}N(N-1)}. \tag{3.1.12}
$$

Hence:

$$
p_{\text{IndGin},2}(\lambda_1,\ldots,\lambda_N)=C_L^{\text{IndGin},2}\frac{\pi^{N^2-N}}{\prod_{j=1}^{N}\Gamma(j)}\prod_{j<k}|\lambda_k-\lambda_j|^2\prod_{j=1}^{N}|\lambda_j|^{2L}e^{-|\lambda_j|^2}. \tag{3.1.13}
$$

Clearly the eigenvalue jpdf is rotationally invariant in its eigenvalues, which means the ordering of the eigenvalues can be lifted by simply dividing the eigenvalue jpdf by the factor $\frac{1}{N!}$. All in all we obtain then:

$$
p_{\text{IndGin},2}(\lambda_1,\ldots,\lambda_N)=C_L^{\text{IndGin},2}\frac{\pi^{N^2-N}}{\prod_{j=1}^{N}j!}\prod_{j<k}|\lambda_k-\lambda_j|^2\prod_{j=1}^{N}|\lambda_j|^{2L}\,\mathrm{e}^{-|\lambda_j|^2}. \tag{3.1.14}
$$

□

Hence the inducing procedure results in the additional factor $\prod_{j=1}^{N}|\lambda_j|^{2L}$ in the symmetrized eigenvalue jpdf of the induced Ginibre ensemble. As a consequence the probability of finding eigenvalues close to zero is small, which result in a repulsion from the origin. The larger the mismatch of dimension in the original rectangular Ginibre matrix, used to generate the complex induced Ginibre ensemble, the stronger the repulsion of eigenvalues away from the origin.

Lemma 3.1.5. *Let $G\in\mathbb{C}^{N\times N}$ be a random matrix pertaining to the complex* *induced Ginibre ensemble. Then its eigenvalue jpdf is correctly normalized by:*

$$c_L^{IndGin} = \frac{1}{\pi^N} \prod_{j=1}^{N} \frac{1}{j\Gamma(j+L)}. \tag{3.1.15}$$

Proof. In order to determine the normalization constant c_L^{IndGin} of the eigenvalue jpdf, we can make use of the Andreief identity [And83]:

Lemma 3.1.6. *[[And83]] Let $(X, d\mu)$ be a measure space and let $f_j, \bar{g}_k \in L^2(X)$ for $1 \leq j, k \leq n$. Then:*

$$\begin{aligned} &\int_X \cdots \int_X \det\left(f_j(x_k)\right)_{1\leq j,k\leq n} \det(\bar{g_j}(x_k))_{1\leq j,k\leq n}\, d\mu(x_1)\cdots d\mu(x_n) \\ &= n!\det\left(\int_X f_j(x)\bar{g}_k(x)d\mu(x)\right)_{1\leq j,k\leq n}. \end{aligned} \tag{3.1.16}$$

We need to calculate:

$$\left(c_L^{\mathrm{IndGin}}\right)^{-1} = \int_{\mathbb{C}^N} \prod_{j<k} |\lambda_k - \lambda_j|^2 \prod_{j=1}^{N} |\lambda_j|^{2L} e^{-|\lambda_j|^2} d^2\lambda_1 \cdots d^2\lambda_N. \tag{3.1.17}$$

It is possible to rewrite the Vandermonde determinant in the following way:

$$\prod_{j<k} |\lambda_k - \lambda_j|^2 = \det\left(\lambda_j^{N-k}\right)_{1\leq j,k\leq N} \det\left(\bar{\lambda}_j^{N-k}\right)_{1\leq j,k\leq N}, \tag{3.1.18}$$

which implies:

$$\left(c_L^{\mathrm{IndGin}}\right)^{-1} = \int_{\mathbb{C}^N} \det\left(\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\lambda_j^{M-k}\right)_{1\leq j,k\leq N} \det\left(\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\bar{\lambda}_j^{M-k}\right)_{1\leq j,k\leq N} d\lambda_1\cdots d\lambda_N.$$

Thus the integral is in the right form for the application of the Andreief identity with:

$$\begin{aligned} f_k(\lambda_j) &= \mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\lambda_j^{M-k} \\ \bar{g}_k(\lambda_j) &= \mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\bar{\lambda}_j^{M-k}. \end{aligned}$$

Consequently:

$$\left(c_L^{\mathrm{IndGin}}\right)^{-1} = N!\det\left(\int_{\mathbb{C}} \mathrm{e}^{-|\lambda|^2}\,\lambda^{M-j}\bar{\lambda}^{M-k} d^2\lambda\right)_{1\leq j,k\leq N}. \tag{3.1.19}$$

As the monomials are orthogonal on the complex plane with respect to the weight w_{IndGin}, the off-diagonal elements of this determinant are zero. For the diagonal entries:

$$\int_{\mathbb{C}} \mathrm{e}^{-|\lambda|^2} |\lambda|^{2(M-j)} d^2\lambda = \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} \mathrm{e}^{-r^2} r^{2(M-j)+1} dr d\phi. = \pi\Gamma(M-j+1). \tag{3.1.20}$$

Thus:

$$\left(c_L^{\mathrm{IndGin}}\right)^{-1} = N!\pi^N \prod_{j=1}^{N} \Gamma(j+L). \tag{3.1.21}$$

□

Setting $L = 0$ one recovers the eigenvalue jpdf for the complex Ginibre ensemble, as already derived in [Gin65] (Ginibre uses the more cumbersome spectral decomposition for the change of variables).

3.1.2 The n-point correlation function and the method of orthogonal polynomials

The correlation functions are defined as follows:

$$R_n(\lambda_1, \ldots, \lambda_n) = \frac{N!}{(N-n)!} \int P(\lambda_1, \ldots, \lambda_N) d\lambda_{n+1} \cdots d\lambda_N. \tag{3.1.22}$$

These are just the marginal density function of the symmetrized eigenvalues jpdf with different normalization constant. They can be related to the statistics of the number $\mathcal{N}_B$ of eigenvalues in the set B:

$$E[\mathcal{N}_B] = \int_B R_1(\lambda) d^2\lambda\,, \tag{3.1.23}$$

$$\mathrm{Var}[\mathcal{N}_B] = E[\mathcal{N}_B] + \int_{B\times B} R_2(\lambda_1, \lambda_2) - R_1(\lambda_1)R_1(\lambda_2) d^2\lambda_1 d^2\lambda_2. \tag{3.1.24}$$

Remark 3.1.7. *For a derivation of those relations, see [Fyo05], chapter 3.*

Relations for higher moments of $\mathcal{N}_B$ involving higher order correlation functions can be similarly deduced. The n-point correlation functions can be compactly derived by employing the method of orthogonal polynomials introduced by Metha and Dyson in the same way as for the complex Ginibre ensemble. At the heart of the method of orthogonal polynomials lies the "integrating-out" lemma.

Lemma 3.1.8. *[[Meh04]] Let $D_n = D_n(X) = (d_{jk})_{1\le j,k\le n}$ denote a $n \times n$ matrix with entries $d_{jk} := f(x_j, x_k)$ for $j, k = 1, \ldots, n$ which depend on the complex vector $X = (x_1, \ldots, x_n)$ and the complex valued function f. If the function f*

satisfies the conditions:

$$\int f(x,y)f(y,z)d\mu(y) = f(x,z) \tag{3.1.25}$$

$$\int f(x,x)d\mu(x) = q\,, \tag{3.1.26}$$

then:

$$\int \det D_n d\mu(x_n) = [q-(n-1)]\det D_{n-1}\,, \tag{3.1.27}$$

where $D_{n-1} = D_{n-1}(\tilde{X})$ with entries $d_{jk} := f(x_j, x_k)$ for $j,k = 1,\ldots,n-1$ which depend on the complex vector $\tilde{X} = (x_1,\ldots,x_{n-1})$.

The first step in retrieving a closed form expression for the n-point correlation function is rewriting the Jacobian in the eigenvalue jpdf using the identity for Vandermonde determinants:

$$\prod_{j<k}|\lambda_k-\lambda_j|^2 = \det\left(\lambda_j^{N-k}\right)_{1\le j,k\le N}\det\left(\bar{\lambda}_j^{N-k}\right)_{1\le j,k\le N}. \tag{3.1.28}$$

This implies for the eigenvalue jpdf:

$$\begin{aligned}
&p_{\mathrm{IndGin},2}(\lambda_1,\ldots,\lambda_N)\\
=&\frac{1}{\pi^N}\prod_{j=1}^{N}\frac{1}{j\Gamma(j+L)}\det\left(\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\lambda_j^{M-k}\right)_{j,k=1}^{N}\det\left(\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\bar{\lambda}_j^{M-k}\right)_{j,k=1}^{N}\\
=&\frac{1}{N!}\det\left(\frac{1}{\sqrt{\pi\Gamma(M-j+1)}}\,\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\lambda_j^{M-k}\right)_{j,k=1}^{N}\times\\
&\det\left(\frac{1}{\sqrt{\pi\Gamma(M-j+1)}}\,\mathrm{e}^{-\frac{1}{2}|\lambda_j|^2}\,\bar{\lambda}_j^{M-k}\right)_{j,k=1}^{N}.
\end{aligned} \tag{3.1.29}$$

The family of monomials $\{p_k\}_{k=0,1,\ldots}$ inside the determinants with: $p_k(\lambda) = \lambda^k$ are now orthogonal on the complex plane with respect to the weight function:

$$w^2_{\mathrm{IndGin},2}(\lambda_j) = |\lambda_j|^{2L}\,\mathrm{e}^{-|\lambda_j|^2}\,. \tag{3.1.30}$$

and with normalization $r_j^{IndGin} = \frac{1}{\pi\Gamma(j+L+1)}$. To make notation easier we set: $A = (a_{jk})_{1\le j,k\le N}$ with $a_{jk} = \frac{1}{\sqrt{r_j}}\,p_j(\lambda_k)\,w_{\mathrm{IndGin},2}\,(\lambda_k)$ for $j,k = 1,\ldots,N$. This leads to:

$$p_{\mathrm{IndGin},2}(\lambda_1,\ldots,\lambda_N) = \frac{1}{N!}\det A\det A^\dagger = \frac{1}{N!}\det\left(\sum_{j=1}^{N}a_{jl}\bar{a}_{jk}\right)_{k,l=1}^{N}. \tag{3.1.31}$$

All in all we can rewrite the eigenvalue jpdf in the following way:

$$p_{\mathrm{IndGin},2}(\lambda_1,\ldots,\lambda_N) = \frac{1}{N!}\det\left(K_N^{\mathrm{IndGin}}(\lambda_k,\lambda_l)\right)_{1\le k,l\le N} d\lambda_1\ldots d\lambda_N\,, \tag{3.1.32}$$

where we have introduced the kernel notation:

$$\begin{aligned} K_N^{\mathrm{IndGin}}(\lambda_k,\lambda_l) &:= w_{\mathrm{IndGin},2}(\lambda_k)w_{\mathrm{IndGin},2}(\lambda_l)\sum_{j=0}^{N-1}\frac{1}{r_j^{\mathrm{IndGin}}}p_j(\lambda_k)p_j(\bar{\lambda}_l) \\ &= \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2}\sum_{j=0}^{N-1}\frac{\left(\lambda_k\bar{\lambda}_l\right)^{j+L}}{\Gamma(j+L+1)}. \end{aligned} \tag{3.1.33}$$

One can easily verify that the kernel $K_N(\lambda_k,\lambda_l)$ satisfies the conditions of lemma 3.1.8. Hence:

$$\begin{aligned} R_{N-1}(\lambda_1,\ldots,\lambda_{N-1}) &= \frac{N!}{[N-(N-1)]!}\int_C \frac{1}{N!}\det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{N} d^2\lambda_N \\ &= \det(K_N(\lambda_j,\lambda_k))_{j,k=1}^{N-1}. \end{aligned} \tag{3.1.34}$$

Repeating this step we obtain:

$$\begin{aligned} &R_{N-2}(\lambda_1,\ldots,\lambda_{N-2}) \\ &= \frac{N!}{[N-(N-2)]!}\int_{\mathbb{C}}\int_{\mathbb{C}}\frac{1}{N!}\det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{N} d^2\lambda_N d^2\lambda_{N-1} \\ &= \frac{1}{[N-(N-2)]!}\int_{\mathbb{C}}\det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{N-1} d^2\lambda_{N-1} \\ &= \det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{N-2}. \end{aligned} \tag{3.1.35}$$

The lower correlation functions can be derived from the higher order ones in the following way:

$$R_n(\lambda_1,\ldots,\lambda_n) = \frac{1}{N-n}\int_{\mathbb{C}} R_{n+1}(\lambda_1,\ldots,\lambda_{n+1})d^2\lambda_{n+1}. \tag{3.1.36}$$

Using induction and applying lemma 3.1.8 $N-n$ times finally yields:

$$\int_{\mathbb{C}}\cdots\int_{\mathbb{C}}\det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{N} d^2\lambda_{n+1}\cdots d^2\lambda_N = \frac{N!}{(N-n)!}\det\left(K_N(\lambda_k,\lambda_l)\right)_{k,l=1}^{n}.$$

Hence we have found a closed form expression for all n-point correlation functions:

Theorem 3.1.9.

$$R_n^{IndGin}(\lambda_1, \ldots, \lambda_n) = \det\left(K_N^{IndGin}(\lambda_k, \lambda_l)\right)_{k,l=1}^n \tag{3.1.37}$$

$$K_N^{IndGin}(\lambda_k, \lambda_l) = w_{IndGin,2}(\lambda_k) w_{IndGin,2}(\lambda_l) \sum_{j=0}^{N-1} \frac{1}{r_j^{IndGin}} p_j(\lambda_k) p_j(\bar{\lambda}_l) \tag{3.1.38}$$

$$= \frac{1}{\pi} e^{-\frac{1}{2}|\lambda_k|^2 - \frac{1}{2}|\lambda_l|^2} \sum_{j=0}^{N-1} \frac{\left(\lambda_k \bar{\lambda}_l\right)^{j+L}}{\Gamma(j+L+1)}. \tag{3.1.39}$$

Especially the mean eigenvalue density is given by:

$$R_1^{\text{IndGin}}(\lambda) = K_N^{\text{IndGin}}(\lambda, \lambda) = \frac{1}{\pi} \mathrm{e}^{-|\lambda|^2} \sum_{j=0}^{N-1} \frac{|\lambda|^{2j+2L}}{\Gamma(j+L+1)} = \rho_N^{\text{IndGin}}(\lambda). \tag{3.1.40}$$

It becomes apparent in the asymptotic analysis of large matrix dimensions that an integral representation of the correlation kernel is beneficial. The advantages of using an integral representation instead of a truncated series are obvious, as it enables the use of the powerful saddle-point or Laplace method. It is possible to express the correlation kernel from equation (3.1.37), theorem 3.1.9 in terms of a difference of incomplete gamma functions.

Lemma 3.1.10.

$$S_{IndGin}(z) = \sum_{j=0}^{N-1} \frac{z^j}{\Gamma(j+L+1)} = \frac{\frac{1}{\Gamma(L)}\gamma(z, L) - \frac{1}{\Gamma(M)}\gamma(z, M)}{z^L \,\mathrm{e}^{-z}} \tag{3.1.41}$$

Proof. The integral representation for $S_{\text{IndGin}}(z)$ can be derived in the following way. Algebraic manipulation of the derivative of $S_{\text{IndGin}}(z)$ show, that $S_{\text{IndGin}}(z)$ satisfies a first-order differential equation. This first order differential equation can be then solved using the method of integrating factors. This then yields an integral representation. Thus note:

$$\begin{aligned} S'_{\text{IndGin}}(z) &= \sum_{j=1}^{N-1} \frac{j z^j}{\Gamma(L+j+1)} = \sum_{j=1}^{N-1} \frac{j}{L+j} \frac{z^{j-1}}{\Gamma(L+j)} \\ &= \sum_{j=1}^{N-1} \left(1 - \frac{L}{L+j}\right) \frac{z^{j-1}}{\Gamma(L+j)} = \sum_{j=1}^{N-1} \frac{z^{j-1}}{\Gamma(L+j)} - L \sum_{j=1}^{N-1} \frac{z^{j-1}}{\Gamma(L+j+1)} \\ &= \sum_{p=0}^{N-2} \frac{z^p}{\Gamma(L+p+1)} - \frac{L}{z} \sum_{j=1}^{N-1} \frac{z^j}{\Gamma(L+j+1)} \\ &= S_{\text{IndGin}}(z) - \frac{z^{N-1}}{\Gamma(M)} - \frac{L}{z} S_{\text{IndGin}}(z) + \frac{L}{z} \frac{1}{\Gamma(L+1)} \end{aligned} \tag{3.1.42}$$

Thus we have obtained the first-order differential equation:

$$S'_{\text{IndGin}}(z) = \left(1 - \frac{L}{z}\right) S_{\text{IndGin}}(z) + \frac{1}{z\Gamma(L)} - \frac{z^{N-1}}{\Gamma(M)} \tag{3.1.43}$$

with boundary condition $S_{\text{IndGin}}(0) = \frac{1}{\Gamma(L+1)}$. We set $h_{\text{IndGin}}(z) = 1 - \frac{L}{z}$ and $c_{\text{IndGin}}(z) = \frac{1}{z\Gamma(L)} - \frac{z^{N-1}}{\Gamma(M)}$ and rewrite:

$$S'_{\text{IndGin}}(z) - h_{\text{IndGin}}(z) S_{\text{IndGin}}(z) = c_{\text{IndGin}}(z).$$

Multiplying both sides with the integrating factor function $I_{\text{IndGin}}(z)$ leads to:

$$I_{\text{IndGin}}(z) c_{\text{IndGin}}(z) = I_{\text{IndGin}}(z) S'_{\text{IndGin}}(z) - I_{\text{IndGin}}(z) h_{\text{IndGin}}(z) S_{\text{IndGin}}(z)$$

This expression is of the form $(hf)' = hf' + h'f$ which gives us:

$$-h_{\text{IndGin}}(z) I_{\text{IndGin}}(z) = I'_{\text{IndGin}}(z). \tag{3.1.44}$$

Consequently:

$$I_{\text{IndGin}}(z) = \mathrm{e}^{-\int_1^z h_{\text{IndGin}}(t)dt} = \mathrm{e}^{-\int_1^z 1 - \frac{L}{t} dt} = z^L \, \mathrm{e}^{-(z-1)} \,. \tag{3.1.45}$$

This implies:

$$[I_{\text{IndGin}}(z) S_{\text{IndGin}}(z)]' = I_{\text{IndGin}}(z) c_{\text{IndGin}}(z). \tag{3.1.46}$$

As a result:

$$\begin{aligned} I_{\text{IndGin}}(z) S_{\text{IndGin}}(z) &= C + \int_0^z I_{\text{IndGin}}(t) c_{\text{IndGin}}(t) dt \\ &= C + \frac{1}{\Gamma(L)} \int_0^z t^{L-1} \, \mathrm{e}^{-(t-1)} \, dt - \frac{1}{\Gamma(M)} \int_0^z t^{M-1} \, \mathrm{e}^{-(t-1)} \, dt \\ &= C + \frac{\mathrm{e}^1}{\Gamma(L)} \gamma(z, L) - \frac{\mathrm{e}^1}{\Gamma(M)} \gamma(z, M) \end{aligned} \tag{3.1.47}$$

Finally:

$$S_{\text{IndGin}}(z) = \frac{C \, \mathrm{e}^{-1} + \frac{1}{\Gamma(L)} \gamma(z, L) - \frac{1}{\Gamma(M)} \gamma(z, M)}{z^L \, \mathrm{e}^{-z}}. \tag{3.1.48}$$

The boundary condition then implies $C = 0$. $\square$

All in all the correlation kernel can be expressed using this integral representation:

$$K_N^{\text{IndGin}}(\lambda_k, \lambda_l) = \frac{1}{\pi} \, \mathrm{e}^{-\frac{1}{2}|\lambda_k|^2 - \frac{1}{2}|\lambda_l|^2 + \lambda_k \bar{\lambda}_l} \left[\frac{\gamma(\lambda_k \bar{\lambda}_l, L)}{\Gamma(L)} - \frac{\gamma(\lambda_k \bar{\lambda}_l, M)}{\Gamma(M)} \right]. \tag{3.1.49}$$

Furthermore the mean eigenvalue density can be written as:

$$\rho_N^{\mathrm{IndGin}}(\lambda) = R_1^{\mathrm{IndGin}}(\lambda) = \frac{1}{\pi}\left[\frac{\gamma(|\lambda|^2, L)}{\Gamma(L)} - \frac{\gamma(|\lambda|^2, M)}{\Gamma(M)}\right]. \tag{3.1.50}$$

Consistency with the result for the complex Ginibre ensemble with $L = 0$ can be shown by using integration by parts in the limit $L \to 0$.

3.1.3 Asymptotic analysis

In the limit of large matrix dimensions it is possible to distinguish two asymptotic regimes: the regime of strong rectangularity, in which the parameter controlling the rectangularity of the matrix grows proportionally with matrix size, and the regime of almost square matrices, in which the rectangularity parameter is kept fixed. In the regime of strong rectangularity, the eigenvalue repulsion from the origin is strong and as a result the mean eigenvalue density is to leading order uniform on an annulus, whose width depends on the rectangularity parameter L. At the circular edges of the mean eigenvalue density one recovers universal behavior, meaning that the same limiting expressions are found as in the complex Ginibre ensemble. Furthermore on the support of the eigenvalue density in the bulk and at the edge of the support the correlation kernel shows universal behavior, again meaning, that in the limit of large matrix dimension the limiting kernels of the complex Ginibre ensemble are recovered.

In the regime of almost square matrices the parameter L is kept fixed and thus the mismatch in dimensions is kept small. As a result the repulsion away from the origin is weak and only creates a microscopically small hole. As a consequence the eigenvalues are to leading order uniformly distributed on the unit disk. In the bulk and at the edge of the eigenvalue support we can again show that the correlation kernels exhibit universal behavior. However, one of the main results of this work is, that at the origin a new correlation kernel emerges in the limit of large matrix dimensions. Indeed it seems that this correlation kernel is universal, in the sense that it can be recovered in different asymptotic regimes for the two additional ensembles studied in this work.

As the eigenvalues of a complex induced Ginibre matrix spread across the whole complex plane for large matrix dimensions, the scaling $\lambda = \sqrt{N}z$ is necessary in the subsequent derivations. This is equivalent to analyzing an ensemble of the form:

$$P(A) \propto \det(AA^\dagger)^L \,\mathrm{e}^{-N\,\mathrm{tr}(AA^\dagger)}\,. \tag{3.1.51}$$

Figure 3.1 shows the eigenvalue distribution of the complex induced Ginibre ensemble in the two asymptotic regimes.

Strongly rectangular limit

In the regime of strong rectangularity, the rectangularity parameter is chosen to grow proportionally to matrix size: $L = N\alpha$, $\alpha > 0$. (This corresponds to the quadratization of 'standing' rectangular matrices of size $(N+L) \times N$.) Starting point of the asymptotic analysis is the mean eigenvalue density $R_1(\lambda)$ from equation (3.1.50).

Theorem 3.1.11.

$$\lim_{N\to\infty} \rho_N^{IndGin}(\sqrt{N}z) = \frac{1}{\pi}\left[\Theta(\sqrt{\alpha+1} - |z|\,) - \Theta(\sqrt{\alpha} - |z|\,)\right]. \tag{3.1.52}$$

Proof. Starting point is the scaled mean density of eigenvalues:

$$\rho_N^{\mathrm{IndGin}}(\sqrt{N}z) = \frac{1}{\pi}\left[\frac{\gamma(N|z|^2, N\alpha)}{\Gamma(N\alpha)} - \frac{\gamma(N|z|^2, N\alpha+N)}{\Gamma(N\alpha+N)}\right]. \tag{3.1.53}$$

Using theorem A.1.1 with $a = \alpha$ and $x = |z|^2$ yields:

$$\lim_{N\to\infty} \frac{1}{\Gamma(N\alpha)}\gamma(N|z|^2, N\alpha) = \Theta(|z| - \alpha), \tag{3.1.54}$$

while applying theorem A.1.1 with $a = \alpha + 1$ and $x = |z|^2$ gives:

$$\lim_{N\to\infty} \frac{1}{\Gamma(N\alpha)}\gamma(N|z|^2, N\alpha) = \Theta(|z| - \alpha). \tag{3.1.55}$$

□

Thus in the limit, when N is large and $L = N\alpha$, the eigenvalue distribution (in the leading order) is uniform and supported by a ring about the origin with the inner and outer radii $r_{in} = \sqrt{L}$ and $r_{out} = \sqrt{L+N}$, respectively. Setting the parameter $L = 0$ one recovers the circular law, theorem 1.2.2. Next we want to estimate how fast the density falls from $\frac{1}{\pi}$ to zero close to the circular edges of the eigenvalue density support.

Theorem 3.1.12. *Close to the circular edges of the eigenvalue support, for every* *angle ϕ:*

$$\lim_{N\to\infty} \rho_N^{IndGin}((r_{out}+\xi)\,\mathrm{e}^{i\phi}) = \lim_{N\to\infty} \rho_N^{IndGin}((r_{in}-\xi)\,\mathrm{e}^{i\phi}) = \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\xi)\ , \quad (3.1.56)$$

Proof. The first edge to be analyzed is the outer circular edge $z = (\sqrt{L+N} + \xi)\,\mathrm{e}^{i\phi}$. Using theorem A.1.1 from appendix A yields:

$$\frac{\gamma(|z|^2, L)}{\Gamma(L)} = \frac{\gamma\big(N(\alpha+1) + 2\sqrt{N(\alpha+1)}\xi + \xi^2), N\alpha\big)}{\Gamma(N\alpha)} \sim 1. \quad (3.1.57)$$

In addition applying theorem A.1.2 from appendix A yields:

$$\begin{aligned}\frac{\gamma(|z|^2, M)}{\Gamma(M)} =& \frac{\gamma\big(N(\alpha+1) + 2\sqrt{N(\alpha+1)}\xi + \xi^2), N(\alpha+1)\big)}{\Gamma\big(N(\alpha+1)\big)} \\ \sim& 1 - \frac{1}{2}\,\mathrm{erfc}(\sqrt{2}\xi).\end{aligned} \quad (3.1.58)$$

All in all we have calculated for the outer edge:

$$\lim_{N\to\infty} \rho_N((r_{out}+\xi)\,\mathrm{e}^{i\phi}) = \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\xi). \quad (3.1.59)$$

Similarly for the inner edge $z = (\sqrt{L+N} + \xi)\,\mathrm{e}^{i\phi}$ using theorem A.1.2 yields:

$$\frac{1}{\Gamma(N\alpha)}\gamma\big(N\alpha - 2\sqrt{N\alpha}\xi + \xi^2), N\alpha\big) \sim \frac{1}{2}\,\mathrm{erfc}(\sqrt{2}\xi), \quad (3.1.60)$$

while applying theorem A.1.1 to:

$$\frac{1}{\Gamma\big(N(\alpha+1)\big)}\gamma\big(N\alpha + 2\sqrt{N\alpha}\xi + \xi^2), N(\alpha+1)\big) \quad (3.1.61)$$

gives:

$$\lim_{N\to\infty} \rho_N^{\mathrm{IndGin}}((r_{in}-\xi)\,\mathrm{e}^{i\phi}) = \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\xi). \quad (3.1.62)$$

□

Noting [AS72], page 298, equation (7.1.23):

$$\mathrm{erfc}(x) \sim \frac{\mathrm{e}^{-|x|^2}}{\sqrt{\pi}|x|}, \quad (3.1.63)$$

the eigenvalue density falls from $\frac{1}{\pi}$ to zero at a Gaussian rate at the inner and outer boundaries of the eigenvalue support. As was observed in equations (42)-(43), [Bog10], the scaling law (3.1.56) belongs to the universality class of the

Feinberg-Zee type ensembles for $\beta = 2$.

The ultimate aim of this work is to investigate universal behavior for the n-point correlation functions of the induced family of random matrix ensembles. Thus in the following the bulk and edge behavior of the n-point correlation functions of the complex induced Ginibre ensemble is analyzed in detail.

Theorem 3.1.13 (The limiting correlation functions in the bulk)**.** *Let $u, z_1, \ldots, z_n$ be complex numbers with $\sqrt{a} \leq |u| \leq \sqrt{a+1}$ and set $\lambda_k = \sqrt{N}u + z_k$ for $k = 1, \ldots, n$ and $L = N\alpha$, then:*

$$\lim_{N\to\infty} R_n^{IndGin}(\lambda_1, \ldots, \lambda_n) = \det\left(\frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|z_k|^2-\frac{1}{2}|z_l|^2+z_l\bar{z}_k}\right)_{k,l=1,}^{n}. \tag{3.1.64}$$

Proof. We start from the integral form of the correlation kernel:

$$K_N^{\mathrm{IndGin}}(\lambda_k, \lambda_l) = \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\left[\frac{\gamma(\lambda_k\bar{\lambda}_l, L)}{\Gamma(L)} - \frac{\gamma(\lambda_k\bar{\lambda}_l, M)}{\Gamma(M)}\right]. \tag{3.1.65}$$

Now note that:

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\\
&= \mathrm{e}^{-\frac{1}{2}N|u|^2-\frac{1}{2}\sqrt{N}(u\bar{z}_k+\bar{u}z_k)-\frac{1}{2}|z_k|^2}\,\mathrm{e}^{-\frac{1}{2}N|u|^2-\frac{1}{2}\sqrt{N}(u\bar{z}_l+\bar{u}z_l)-\frac{1}{2}|z_k|^2}\,\mathrm{e}^{N|u|^2+\sqrt{N}(u\bar{z}_l+\bar{u}z_k)-z_k\bar{z}_l}\\
&\sim \mathrm{e}^{-\frac{1}{2}|z_k|^2-\frac{1}{2}|z_l|^2+z_k\bar{z}_l}\,.
\end{aligned} \tag{3.1.66}$$

Furthermore applying theorem A.1.3 gives the desired result. □

Finally the remaining task is to compute the large N limit for the correlation functions at the edge. As opposed to the Ginibre ensemble we again have two circular edges to consider.

Theorem 3.1.14 (The limiting correlation functions at the edges)**.** *Let $u, z_1, \ldots, z_n$ be complex numbers with $|u| = 1$, setting $\lambda_k = \sqrt{N(\alpha+1)}u + z_k$ for $k = 1, \ldots, n$ leads to the limiting correlation functions at the outer edge $r_{out} = \sqrt{L+N}$:*

$$\lim_{N\to\infty} R_n^{IndGin}(\lambda_1, \ldots, \lambda_n) = \det\left[\frac{1}{2\pi}\,\mathrm{e}^{-\frac{1}{2}|z_j|^2-\frac{1}{2}|z_k|^2+z_j\bar{z}_k}\,\mathrm{erfc}\left(\frac{z_j\bar{u}+\bar{z}_k u}{\sqrt{2}}\right)\right]_{j,k=1}^{n}.$$

The same limiting expression is found around the inner edge $r_{in} = \sqrt{L}$ of the eigenvalue density by setting $\lambda_k = \sqrt{N\alpha}u - z_k$ for $k = 1, \ldots, n$.

Proof. Let us start with the outer edge r_{out}. Again we start from the integral representation of the kernel:

$$K_N^{\mathrm{IndGin}}(\lambda_k, \lambda_l) = \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\left[\frac{\gamma(\lambda_k\bar{\lambda}_l, L)}{\Gamma(L)} - \frac{\gamma(\lambda_k\bar{\lambda}_l, M)}{\Gamma(M)}\right]. \tag{3.1.67}$$

and note that:

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\\
=\,&\mathrm{e}^{-\frac{1}{2}N(\alpha+1)|u|^2-\frac{1}{2}\sqrt{N(\alpha+1)}(u\bar{z}_k+\bar{u}z_k)-\frac{1}{2}|z_k|^2}\,\mathrm{e}^{-\frac{1}{2}N(\alpha+1)|u|^2-\frac{1}{2}\sqrt{N(\alpha+1)}(u\bar{z}_l+\bar{u}z_l)-\frac{1}{2}|z_k|^2}\\
&\mathrm{e}^{N(\alpha+1)|u|^2+\sqrt{N(\alpha+1)}(u\bar{z}_l+\bar{u}z_k)-z_k\bar{z}_l}\\
\sim\,&\mathrm{e}^{-\frac{1}{2}|z_k|^2-\frac{1}{2}|z_l|^2+z_k\bar{z}_l}\,.
\end{aligned} \tag{3.1.68}$$

Now applying theorem A.1.4 from appendix A yields the desired result. The inner edge result is derived analogously. □

Consequently we have shown that the correlation functions of the complex induced Ginibre ensemble show universal behavior in the bulk and at the edge of the eigenvalue support, meaning that the limiting correlation kernels for the induced complex Ginibre ensemble coincide with the limiting correlation kernels of the complex Ginibre ensemble, see [BS09] appendix C.

Almost square matrices

Now we shall explore a different regime when the rectangularity index $L = M - N \ll N$. This corresponds to the quadratization of almost square matrices. In the vicinity of the origin the corresponding large-N (or equivalently large-M) limit can be performed by simply extending the summation in (3.1.37) to infinity. For the mean eigenvalue density, $\rho_N(\lambda) = R_1(\lambda)$ it gives:[1]

$$\lim_{N\to\infty} \rho_N^{\mathrm{IndGin}}(\lambda) = \frac{1}{\pi} e^{-|\lambda|^2} \sum_{j=0}^{\infty} \frac{|\lambda|^{2(j+L)}}{\Gamma(j+L+1)} = \frac{1}{\pi}\frac{\gamma(L, |\lambda|^2)}{\Gamma(L)}\,. \tag{3.1.69}$$

and more generally:

$$\lim_{N\to\infty} R_n^{\mathrm{IndGin}}(\lambda_1, \ldots, \lambda_n) = \det\left(K_{\mathrm{origin}}^{\mathrm{IndGin}}(\lambda_k, \lambda_l)\right)_{k,l=1}^{n}\,, \tag{3.1.70}$$

[1] When $L = 1$ or $L = 2$ the limiting density can be obtained from the solution [Ake01] of a different random matrix ensemble

with

$$K^{\mathrm{IndGin}}_{\mathrm{origin}}(\lambda_k,\lambda_l) = \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\,\frac{\gamma(L,\lambda_k\bar{\lambda}_l)}{\Gamma(L)}$$
$$= \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\,\frac{1}{\Gamma(L)}\int_0^{\lambda_k\bar{\lambda}_l} t^{L-1}\,\mathrm{e}^{-t}\,dt\ . \tag{3.1.71}$$

At the origin the eigenvalue density vanishes algebraically, $\rho_N(\lambda)\sim\frac{1}{\pi}\frac{|\lambda|^{2L}}{\Gamma(L+1)}$ as $\lambda\to 0$, uniformly in N. Away from the origin, the density reaches its asymptotic value $1/\pi$ very quickly[2]. This plateau extends to a full circle of radius $\sqrt{N}$:

$$\lim_{N\to\infty}\rho_N^{\mathrm{IndGin}}(\sqrt{N}z) = \frac{1}{\pi}\,\Theta(1-|z|)\ , \tag{3.1.72}$$

as in the Ginibre ensemble, and, moreover, for reference points $\sqrt{N}u$, $|u|<1$, one also recovers the Ginibre correlations.

Another quantity of interest is the so-called hole probability $A(s)$ at the origin giving the probability that no eigenvalues lies inside the disk $D_s=\{z:|z|<s\}$. For finite N the hole probability $A^{\mathrm{IndGin}}(s)$ can be derived from the expression:

$$A^{\mathrm{IndGin}}(s) = \int P(\lambda_1,\ldots,\lambda_N)\prod_{j=1}^{N}(1-\chi_{D_s}(\lambda_j))d^2\lambda_1\ldots d^2\lambda_N, \tag{3.1.73}$$

where χ_{D_s} denotes the indicator function of D_s by employing the method of orthogonal polynomials to yield:

$$A^{\mathrm{IndGin}}(s) = \prod_{j=1}^{N}\frac{\Gamma(j+L,s^2)}{\Gamma(j+L)}. \tag{3.1.74}$$

In the asymptotic regime of almost square matrices taking the large N limit, while keeping L fixed, results in the easily accessible expression for the hole probability $A^{\mathrm{IndGin}}(s)=1-\frac{s^{2(L+1)}}{(L+1)!}+O(\frac{s^{2(L+2)}}{(L+2)!})$.

3.1.4 Summary of results

- The eigenvalue jpdf of a complex induced Ginibre matrix:

$$p_{\mathrm{IndGin},2}(\lambda_1,\ldots,\lambda_N) \propto \prod_{j<k}|\lambda_k-\lambda_j|^2\prod_{j=1}^{N}|\lambda_j|^{2L}\,\mathrm{e}^{-|\lambda_j|^2}\ . \tag{3.1.75}$$

[2] A similar behavior is found in the chiral Ginibre ensemble, see [APS09b] for a discussion in the context of gap probabilities.

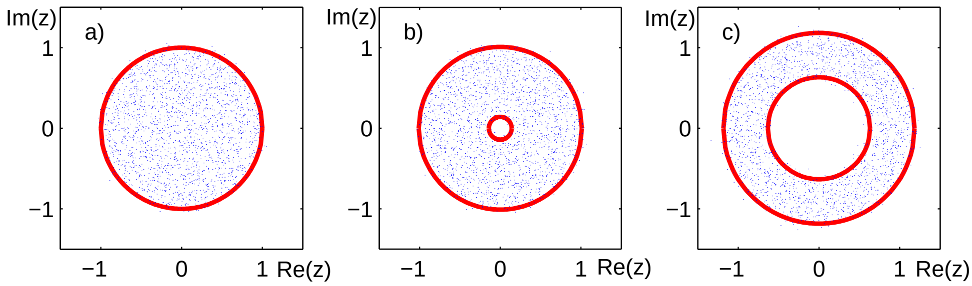


Figure 3.1: Spectra of matrices pertaining to the induced Ginibre ensemble of complex matrices for dimension $N = 100$ and a) $L = 0$, b) $L = 2$, c) $L = 40$. Each plot consists of data from 20 independent realizations. The spectra are rescaled by a factor of $1/\sqrt{L+N}$ and the circles of radius $r_{\text{in}} = \sqrt{L/(L+N)}$ (inner one) and $r_{\text{out}} = 1$ (outer one) are depicted to guide the eye.

- The finite N mean eigenvalue density of a complex induced Ginibre matrix:

$$\rho_N^{\text{IndGin}}(\lambda) = \frac{1}{\pi}\left[\frac{\gamma(|\lambda|^2, L)}{\Gamma(L)} - \frac{\gamma(|\lambda|^2, M)}{\Gamma(M)}\right]. \tag{3.1.76}$$

- Limiting mean eigenvalue density in the regime of strong rectangularity, bulk and edge behavior:

$$\begin{aligned} \lim_{N\to\infty} \rho_N^{\text{IndGin}}(\sqrt{N}z) &= \frac{1}{\pi}\left[\Theta(\sqrt{\alpha+1} - |z|\,) - \Theta(\sqrt{\alpha} - |z|\,)\right] \\ \lim_{N\to\infty} \rho_N^{\text{IndGin}}((r_{out} + \xi)\,\mathrm{e}^{i\phi}) &= \lim_{N\to\infty} \rho_N^{\text{IndGin}}((r_{in} - \xi)\,\mathrm{e}^{i\phi}) \\ &= \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\xi). \end{aligned} \tag{3.1.77}$$

- Limiting mean eigenvalue density in the regime of almost square matrices, bulk and edge behavior:

$$\begin{aligned} \lim_{N\to\infty} \rho_N^{\text{IndGin}}(\sqrt{N}z) &= \frac{1}{\pi}\,\Theta(1 - |z|\,) \\ \lim_{N\to\infty} \rho_N^{\text{IndGin}}((\sqrt{N} + \xi)\,\mathrm{e}^{i\phi}) &= \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\xi). \end{aligned} \tag{3.1.78}$$

- Limiting correlation kernel in the bulk in the regime of strong rectangularity: complex Ginibre, see theorem 3.1.13.

- Limiting correlation kernel in the bulk in the regime of almost square matrices: complex Ginibre, see theorem 3.1.13.
  Limiting correlation kernel at the origin in the regime of almost square

matrices:

$$K^{\rm IndGin}_{\rm origin}(\lambda_k,\lambda_l)=\frac{1}{\pi}\,{\rm e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\,\frac{\gamma(L,\lambda_k\bar{\lambda}_l)}{\Gamma(L)}. \tag{3.1.79}$$

3.2 The complex induced Jacobi and the complex induced spherical ensemble

In the following chapter two further examples of complex induced random matrix ensembles will be presented: the complex induced spherical ensemble and the complex induced Jacobi ensemble. We proceed to deriving the eigenvalue jpdf of the new induced ensembles and computing the $n-$point correlation functions using the method of orthogonal polynomials. An extensive asymptotic analysis is then undertaken. An important result of this asymptotic analysis is the discovery of a new type of limiting correlation kernel in the regime of strong rectangularity and partially weak non-unitarity for the induced Jacobi ensemble.

3.2.1 The induced spherical ensemble: Eigenvalue jpdf

Applying the inducing procedure to a matrix Y pertaining to the complex rectangular spherical ensemble yields a random matrix A pertaining to the complex induced spherical ensemble.

Definition 3.2.1. *The complex induced spherical ensemble with parameters n, M is specified by the following probability measure on the space of $N\times N$ matrices: ${\rm d}\,\mu^{Induced}_{Spherical,2}(G)=P^{Induced}_{Spherical,2}(G)(dG)$, with*

$$P^{Induced}_{Spherical,2}(G)=C^{IndSpherical,2}_{M,N,n}\frac{\det(GG^\dagger)^{M-N}}{\det(I+GG^\dagger)^{n+M}}\quad M\geq N. \tag{3.2.1}$$

In the following we set $L=M-N$. Setting the parameter $L=0$ gives a square matrix pertaining to the matrix-variate t-distribution from definition 2.0.24. Again the parameter L controls the mismatch of dimensions of the rectangular matrix from definition 2.0.24, which is used to generate the induced spherical matrix. Additionally $n-N$ denotes the mismatch of dimensions of the rectangular matrix, which is used to generate the Wishart matrix in definition 2.0.24. The subsequent analysis is again valid for a more general set-up, replacing the integer parameters L and $n+M$ with positive real parameters p,q. However in this case the matrix interpretation for the probability density in equation (4.2.1) is lost.

Lemma 3.2.2. *The element jpdf of a complex induced spherical matrix is correctly normalized using:*

$$C_{M,N,n}^{\mathit{IndSpherical},2} = \pi^{-N^2} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(n+L+j)}{\Gamma(L+j)\Gamma(n-N+j)}. \tag{3.2.2}$$

Proof. The normalization constant is determined by using the singular value decomposition of $G = U\Sigma V^\dagger$ in order to perform integration over the element joint probability density function.

$$\begin{aligned}
\left(C_{M,N,n}^{\text{IndSpherical},2}\right)^{-1} &= \int_{(G)} \frac{\det(GG^\dagger)^L}{\det(I+GG^\dagger)^{n+M}}(dG) && (3.2.3)\\
&= \mathrm{Vol}(U(N))\,\mathrm{Vol}(U[N]) \int_{(\Sigma)} \prod_{j<k}(\sigma_j^2-\sigma_k^2)^2 \prod_{j=1}^{N} \frac{\sigma_j^{2L}}{(1+\sigma_j^2)^{n+M}}(d\Sigma).
\end{aligned}$$

The integral can be further simplified using the already proven results for the volume of the unitary group and applying a simple change of variables:

$$\begin{aligned}
&\left(C_{M,N,n}^{\text{IndSpherical},2}\right)^{-1}\\
=&\frac{\pi^{N^2}}{N!\prod_{j=1}^{N}\Gamma^2(j)} \int_0^\infty \cdots \int_0^\infty \prod_{j<k}(s_j-s_k)^2 \prod_{j=1}^{N} \frac{s_j^{L-\frac{1}{2}}}{(1+s_j)^{n+M}} ds_1 \ldots ds_N. \qquad (3.2.4)
\end{aligned}$$

The factor $\frac{1}{N!}$ is introduced by removing the ordering of the singular values. The change of variables $s_j = \frac{t_j}{1-t_j}$ transforms I into a Selberg integral which can then be evaluated using theorem D.1.1. □

We can now use our knowledge of the element jpdf to derive the jpdf for the eigenvalues of a complex induced spherical random matrix. The derivation of the eigenvalue jpdf follows the method applied in [Kri09, For10b].

Theorem 3.2.3. *The eigenvalue jpdf of a random matrix $G \in \mathbb{C}^{N\times N}$ pertaining to the complex induced spherical ensemble is given by:*

$$p_{\mathit{IndSpherical},2}(\lambda_1,\ldots,\lambda_N) = c_{M,N,n}^{\mathit{IndSpherical}} \prod_{j<k}|\lambda_k-\lambda_j|^2 \prod_{j=1}^{N} \frac{|\lambda_j|^{2L}}{(1+|\lambda_j|^2)^{n+L+1}}. \tag{3.2.5}$$

with

$$c_{M,N,n}^{\mathit{IndSpherical}} = \frac{1}{N!\pi^N} \prod_{j=0}^{N-1} \frac{\Gamma(L+j+1)\Gamma(n-N+j+1)}{\Gamma(N+L+1)}. \tag{3.2.6}$$

Proof. Again the starting point of the derivation is the complex Schur decomposition, lemma 1.3.12: $G = URU^\dagger$ where $U \in \mathbb{C}^{N\times N}$ is an unitary matrix,

$R = \Lambda + S$ with $S \in \mathbb{C}^{N\times N}$ a strictly upper triangular matrix and Λ a diagonal matrix containing eigenvalues of G. The eigenvalue jpdf can now be obtained by integrating out the auxiliary variables U and S.

$$\begin{aligned} &p_{\text{IndSpherical},2}(\lambda_1,\dots,\lambda_N) \\ =&\frac{\pi^{\frac{1}{2}N(N-1)}C^{\text{IndSpherical},2}_{M,N,n}}{\prod_{j=1}^{N}\Gamma(j)}\prod_{j<k}|\lambda_k-\lambda_j|^2\int_{(S)}\frac{\det(RR^\dagger)^L}{\det(I_N+RR^\dagger)^{n+M}}(dS) \end{aligned} \tag{3.2.7}$$

Noting:

$$\det\big(RR^\dagger\big)^L = \prod_{j=1}^{N}|\lambda_j|^{2L} \quad \text{and} \quad \det\big(I_N + (URU^\dagger)(URU^\dagger)^\dagger\big) = \det\big(I_N + RR^\dagger\big),$$

yields:

$$\begin{aligned} p_{\text{IndSpherical},2}(\lambda_1,\dots,\lambda_N) = \frac{\pi^{-\frac{1}{2}N(N+1)}}{\prod_{j=1}^{N}\Gamma(j+1)}\prod_{j=0}^{N-1}\frac{\Gamma(L+j+1)\Gamma(n-N+j+1)}{\Gamma(n+L+j+1)}\times \\ \prod_{j<k}|\lambda_k-\lambda_j|^2\prod_{j=1}^{N}|\lambda_j|^{2L}\int_{(S)}\det(I_N+RR^\dagger)^{-n-M}(dS). \end{aligned} \tag{3.2.8}$$

The integration over the triangular part of R can be performed by introducing a recurrence relation for the following integral:

$$I_{k,M,n} := \int_{(S_k)}\det\big(I_N + R_kR_k^\dagger\big)^{-n-M}(dS_k), \quad k \le N\,,$$

where $R_k = (\Lambda_k + S_k) \in \mathbb{C}^{k\times k}$ is triangular and S_k denotes its strictly upper triangular part, while $\Lambda_k = \mathrm{diag}(\lambda_1,\dots,\lambda_k)$. The sub script $_k$ denotes the matrix dimension. It is then possible to write:

$$I_k + R_kR_k^\dagger = \begin{pmatrix} I_{k-1}+R_{k-1}R_{k-1}^\dagger & \bar{\lambda}_k\vec{u}_{k-1} \\ \lambda_k\vec{u}_{k-1}^\dagger & 1+|\lambda_k|^2 \end{pmatrix}. \tag{3.2.9}$$

Using the block determinant formula:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det\big(A - BD^{-1}C\big), \tag{3.2.10}$$

as well as noting $1 - \frac{|\lambda_k|^2}{1+|\lambda_k|^2} = \frac{1}{1+|\lambda_k|^2}$ yields:

$$\det\big(I_k + R_k R_k^\dagger\big) = (1+|\lambda_k|^2)\det\big(I_{k-1} + R_{k-1}R_{k-1}^\dagger + \frac{\vec{u}_{k-1}\vec{u}_{k-1}^\dagger}{1+|\lambda_k|^2}\big). \tag{3.2.11}$$

Furthermore:

$$\begin{aligned}\det\big(I_k + R_k R_k^\dagger\big) =&(1+|\lambda_k|^2)\det\big(I_{k-1} + R_{k-1}R_{k-1}^\dagger\big) \times \\ &\Big(1 + \frac{1}{1+|\lambda_k|^2}\vec{u}_{k-1}^\dagger\big(I_k + R_k R_k^\dagger\big)^{-1}\vec{u}_{k-1}\Big)\end{aligned} \tag{3.2.12}$$

As a result:

$$\begin{aligned}I_{k,M,n} = (1+|\lambda_k|^2)^{-n-M}&\int_{(S_{k-1})}\det\big(I_{k-1} + R_{k-1}R_{k-1}^\dagger\big)^{-n-M} \times \\ &\Big(1 + \frac{1}{1+|\lambda_k|^2}\vec{u}_{k-1}^\dagger\big(I_k + R_k R_k^\dagger\big)^{-1}\vec{u}_{k-1}\Big)^{-n-M}(dS_{k-1}).\end{aligned} \tag{3.2.13}$$

A change of variables $\vec{v}_{k-1} = (1+|\lambda_k|^2)^{-\frac{1}{2}}\big(I_{k-1}+R_{k-1}R_{k-1}^\dagger\big)^{-\frac{1}{2}}\vec{u}_{k-1}$ with Jacobian $(d\vec{u}_{k-1}) = (1+|\lambda_k|^2)^{N-1}\det\big(I_{k-1} + R_{k-1}R_{k-1}^\dagger\big)(d\vec{v}_{k-1})$ then leads to:

$$I_{k,M,n} = (1+|\lambda_k|^2)^{-n-L-1}\int_{(\vec{v}_{k-1})}\big(1 + (\vec{v}_{k-1})^\dagger(\vec{v}_{k-1})\big)^{-n-M}(d\vec{v}_{k-1})\; I_{k-1,M-1,n}.$$

In addition set:

$$Q_{k-1,M,n} := \int_{(\vec{v}_{k-1})}\big(1 + (\vec{v}_{k-1})^\dagger(\vec{v}_{k-1})\big)^{-n-M}(d\vec{v}_{k-1}). \tag{3.2.14}$$

The above approach can be iterated:

$$\begin{aligned}I_{N,M,n} &= (1+|\lambda_k|^2)^{-L-n-1}(1+|\lambda_{N-1}|^2)^{-L-n-1}\,Q_{N-1,M,n}\,Q_{N-2,M,n}\,I_{N-2,M-2,n} \\ &= \prod_{j=0}^{N-1}\frac{Q_{N-j,M-j,n}}{(1+|\lambda_k|^2)^{n+L+1}}.\end{aligned} \tag{3.2.15}$$

As a consequence:

$$p_{\text{IndSpherical},2}(\lambda_1,\ldots,\lambda_N) = c_{M,N,n}^{\text{IndSpherical}}\prod_{j<k}|\lambda_k - \lambda_j|^2\prod_{j=1}^{N}\frac{|\lambda_j|^{2L}}{(1+|\lambda_k|^2)^{n+L+1}}. \tag{3.2.16}$$

The normalization constant can again be determined through:

$$\big(c_{M,N,n}^{\text{IndSpherical}}\big)^{-1} = \int_{(\Lambda)}\prod_{j<k}|\lambda_k - \lambda_j|^2\prod_{j=1}^{N}\frac{|\lambda_j|^{2L}}{(1+|\lambda_k|^2)^{n+L+1}}(d\Lambda). \tag{3.2.17}$$

Rewriting the Vandermonde gives:

$$\begin{aligned}&\left(c_{M,N,n}^{\text{IndSpherical}}\right)^{-1}\\ &=\int_{(\Lambda)}\det\Big(\frac{\lambda_j^{M-k}}{(1+|\lambda_k|^2)^{\frac{n+L+1}{2}}}\Big)_{j,k=1}^{N}\det\Big(\frac{\bar{\lambda}_j^{M-k}}{(1+|\lambda_k|^2)^{\frac{n+L+1}{2}}}\Big)_{j,k=1}^{N}(d\Lambda),\end{aligned}\tag{3.2.18}$$

and using the Andreief identity, from lemma 3.1.6 yields:

$$\left(c_{M,N,n}^{\text{IndSpherical}}\right)^{-1}=N!\det\Big(\int_{\mathbb{C}}\frac{\lambda^{M-j}\bar{\lambda}^{M-k}}{(1+|\lambda_k|^2)^{n+L+1}}d^2\lambda\Big)_{j,k=1}^{N}.\tag{3.2.19}$$

The monomials are orthogonal with respect to the weight $w^2_{\text{IndSpherical},2}(\lambda)=(1+|\lambda_k|^2)^{-n-L-1}$ on the complex plane. Thus:

$$\begin{aligned}\left(c_{M,N,n}^{\text{IndSpherical}}\right)^{-1}&=N!\prod_{j=1}^{N}\int_{\mathbb{C}}\frac{|\lambda|^{2(M-j)}}{(1+|\lambda_k|^2)^{n+L+1}}d^2\lambda\\ &=N!\prod_{j=1}^{N}\pi\int_0^\infty\frac{R^{M-j}}{(1+R)^{n+L+1}}dR\\ &=N!\pi^N\prod_{j=1}^{N}\int_0^1 s^{M-j}(1-s)^{n+L+1}ds\\ &=N!\pi^N\prod_{j=0}^{N-1}\frac{\Gamma(L+j+1)\Gamma(n-N+j+1)}{\Gamma(n+L+1)}.\end{aligned}\tag{3.2.20}$$

□

Setting the parameters $n=M=N$ one recovers the eigenvalue jpdf of the complex spherical ensemble, which was calculated in [FK09]. The eigenvalue jpdf of the induced spherical ensemble differs from the eigenvalue jpdf of the spherical ensemble by the factor $\prod_{j=1}^N|\lambda|^{2L}$, which is introduced by the inducing procedure. Again the probability of finding eigenvalues close to the origin is small, as eigenvalues are repulsed from the origin. The strength of repulsion is controlled by the rectangularity parameter L.

3.2.2 The induced Jacobi ensemble: Eigenvalue jpdf

Applying the inducing procedure to a random rectangular truncation $A\in\mathbb{C}^{M\times N}$ yields a random matrix G pertaining to the complex induced Jacobi ensemble.

Definition 3.2.4. *For $K\geq M+N$ the complex induced Jacobi ensemble with parameters K,M is specified by the following probability measure on the space of*

$N \times N$ matrices: $\mathrm{d}\,\mu^{Induced}_{Jacobi,2}(G) = P^{Induced}_{Jacobi,2}(G)(dG)$*, with*

$$P^{Induced}_{Jacobi,2}(G) = \gamma^{IndJacobi,2}_{K,M,N} \det(GG^{\dagger})^{M-N} \det\big(I_N - GG^{\dagger}\big)^{K-M-N}. \tag{3.2.21}$$

In the following we set $L = M - N$. Setting the parameter $L = 0$ one recovers the matrix measure of truncations of random unitary matrices [SŻ00, For06] . Note that the parameters:

$$l_M := K - M \qquad l_N := K - N\,, \tag{3.2.22}$$

denote the number of rows l_M and columns l_N, that are deleted from the initial unitary matrix used to generate the ensemble. The name of the ensemble refers to the fact, that the induced measure, see equation (3.2.21) boast a Jacobi weight. For $K < N + M$ the matrix measure of the induced Jacobi ensemble contains δ functions and thus is singular.

Lemma 3.2.5. *For $K \geq M + N$ the induced Jacobi ensemble is correctly normalized using:*

$$\gamma^{IndJacobi,2}_{K,M,N} = \pi^{-N^2} \prod_{j=1}^{N} \frac{\Gamma(K-N+j)\Gamma(j)}{\Gamma(L+j)\Gamma(K-M-N+j)}. \tag{3.2.23}$$

Proof. We need to compute:

$$\big(\gamma^{\mathrm{IndJacobi,2}}_{K,M,N}\big)^{-1} = \int_{(A)} \det(AA^{\dagger})^{L} \det\big(I_N - AA^{\dagger}\big)^{K-M-N}(dA). \tag{3.2.24}$$

Start by changing variables $AA^{\dagger} = W$ with Jacobian:

$$|J| = 2^{-N}\,\mathrm{Vol}\big(U(N)\big) = \frac{\pi^{\frac{1}{2}N(N+1)}}{\prod_{j=1}^{N}\Gamma(j)}. \tag{3.2.25}$$

Then:

$$\big(\gamma^{\mathrm{IndJacobi,2}}_{K,M,N}\big)^{-1} = \frac{\pi^{\frac{1}{2}N(N+1)}}{\prod_{j=1}^{N}\Gamma(j)} \int_{(W)} \det(W)^{L} \det\big(I_N - W\big)^{K-M-N}(dW). \tag{3.2.26}$$

Now change variables to the eigenvalues $x_1, \ldots, x_N$ of W:

$$\left(\gamma_{K,M,N}^{\text{IndJacobi},2}\right)^{-1} = \frac{\pi^{N^2}}{N! \prod_{j=1}^{N} \Gamma^2(j)} \times$$
$$\int_0^1 \cdots \int_0^1 \prod_{1 \le i \le j \le N} |x_i - x_j|^2 \prod_{j=1}^{N} x_j^L (1 - x_j)^{K-N-M} dx_1 \cdots dx_N. \tag{3.2.27}$$

Using the Selberg integral formula from theorem D.1.1 gives the desired result. □

We thus move on to analyzing the eigenvalue distribution of complex induced Jacobi matrices. Note that for the complex induced Jacobi measure (as for its counterpart the square truncations of unitary matrices) the matrix measure only exists if $K \geq N+M$, meaning that, a sufficient number of rows and columns need to be deleted from the unitary matrix used to generate the complex induced Jacobi matrix. Nevertheless, even though the matrix measure is singular for $K < M+N$, it is still possible to derive the distribution of eigenvalues for all possible values of K, M, N. In order to avoid the singularity of the matrix measure in the derivation of the eigenvalue jpdf, we start with the joint distribution of the matrices A, C from theorem 2.0.28. Then using the quadratization from chapter 2 a change of variable is applied, such that we arrive at the joint distribution of G, W, C. Here G denotes the square quadratization of A. Incidently G is a complex induced Jacobi matrix and by using the Schur decomposition and integrating out W as well as C it is possible to derive the eigenvalue jpdf of an induced Jacobi matrix for all possible integer values of K, M, N.

Theorem 3.2.6. *Let $G \in \mathbb{C}^{N\times N}$ be a random matrix pertaining to the complex induced Jacobi ensemble. Then its eigenvalue jpdf is given by:*

$$p_{IndJacobi,2}(\lambda_1, \ldots, \lambda_N) = c_{K,M,N}^{IndJacobi} \prod_{j<k} |\lambda_k - \lambda_j|^2 \prod_{j=1}^{N} |\lambda_j|^{2L} (1 - |\lambda_j|^2)^{l_M - 1}, \tag{3.2.28}$$

where

$$c_{K,M,N}^{IndJacobi} = \frac{1}{\pi} \prod_{j=1}^{N} \frac{\Gamma(l_N + j)}{\Gamma(l_M)\Gamma(L + j)}. \tag{3.2.29}$$

Proof. As the element jpdf for $K < M + N$ is singular, starting poing of our derivation is the joint distribution of the truncations A, C:

$$P(A, C) = c_{\text{Stief}} \delta\left(A^\dagger A + C^\dagger C - I_N\right). \tag{3.2.30}$$

We apply the quadratization procedure by changing variables form the rectangular matrix $A = \begin{pmatrix} Y \\ Z \end{pmatrix}$ to W, G where $W^\dagger A = \begin{pmatrix} G \\ O \end{pmatrix}$ and thus $A = W\begin{pmatrix} G \\ 0 \end{pmatrix}$. The matrix W is unitary and the decomposition is unique if W is chosen from the coset $U(M)/\big(U(N) \times U(M-N)\big)$. The Jacobian of this change of variables is then given by:

$$|J| = \det\left(GG^\dagger\right)^L. \tag{3.2.31}$$

Furthermore note:

$$A^\dagger A = \begin{pmatrix} G & O \end{pmatrix} W^\dagger W \begin{pmatrix} G \\ O \end{pmatrix} = G^\dagger G. \tag{3.2.32}$$

As a result:

$$P(W, G, C) = c_{\text{Stief},2} \det\left(GG^\dagger\right)^L \delta\big(G^\dagger G + C^\dagger C - I_N\big). \tag{3.2.33}$$

Integrating out the matrix W then yields:

$$P(G, C) = c_{\text{Stief},2} \frac{\mathrm{Vol}(U(M))}{\mathrm{Vol}(U(L))\,\mathrm{Vol}(U(N))} \det\left(GG^\dagger\right)^L \delta\big(G^\dagger G + C^\dagger C - I_N\big). \tag{3.2.34}$$

We can now employ the complex Schur decomposition from theorem 1.3.12 $G = V(\Lambda + S)V^\dagger$ and obtain after integrating out V:

$$\begin{aligned} P(\Lambda, S, C) = c_{\text{Stief}} &\frac{\mathrm{Vol}(U(M))}{\mathrm{Vol}(U(L))\,\mathrm{Vol}(U(N))} \mathrm{Vol}(U[N]) \\ &\prod_{j<k} |\lambda_j - \lambda_k|^2 \det\left(\Lambda\Lambda^\dagger\right)^L \delta\big((\Lambda^\dagger + S^\dagger)(\Lambda + S) + C^\dagger C - I_N\big). \end{aligned} \tag{3.2.35}$$

We need to make use of the following lemma:

Lemma 3.2.7.

$$\begin{aligned} &\int_{(S)} \int_{(C)} \delta\big((\Lambda^\dagger + S^\dagger)(\Lambda + S) + C^\dagger C - I_N\big)(dC)(dS) \\ &= \left[\frac{\mathrm{Vol}(U(l_M))}{\mathrm{Vol}(U(l_M - 1))}\right]^N \prod_{j=1}^{N} (1 - |\lambda_j|^2)^{l_M - 1} \end{aligned} \tag{3.2.36}$$

Proof. It is helpful to divide the matrix C into N columns of size $l_M \times 1$, $C = \big(C_1, \ldots, C_N\big)$. The proof of this lemma is inspired from [KSŻ10, SŻ00], though it varies in details. The idea is to first integrate out the upper block-triangular matrix S, by integrating out each of its entries, starting from the leftmost entry in the first row and then moving row by row. Now the delta function gives the following conditions on the matrix entries for $j = 1, \dots, N$ and $k = j+1, \dots, N$:

$$\bar{\lambda}_j S_{jk} + C_j^\dagger C_k + \sum_{l<j} \bar{S}_{lj} S_{lk} = 0 \tag{3.2.37}$$

$$|\lambda_j|^2 + C_j^\dagger C_j + \sum_{l<j} |S_{lj}|^2 - 1 = 0. \tag{3.2.38}$$

Especially the first row gives for $k = 2, \dots, N$:

$$\bar{\lambda}_1 S_{1k} + C_1^\dagger C_k = 0. \tag{3.2.39}$$

The first step is changing variables for $k = 2, \dots, N$:

$$S_{1k}^{(1)} = \bar{\lambda}_1^{-1} S_{1j} \tag{3.2.40}$$

with Jacobian:

$$|J_1| = \prod_{k=2}^{N} |\lambda_1|^{-2}. \tag{3.2.41}$$

Note that $S_{1k}^{(1)} = -C_1^\dagger C_k$. Using this relation for $j = 2, \dots, N$ yields:

$$\begin{aligned} &|\lambda_j|^2 + |\lambda_1|^{-2} C_j^\dagger C_1 C_1^\dagger C_j + C_j^\dagger C_j + \sum_{1<l<j} \bar{S}_{lj} S_{lj} - 1 = 0 \\ \Leftrightarrow\ &|\lambda_j|^2 + C_j^\dagger \big(|\lambda_1|^{-2} C_1 C_1^\dagger + I_{l_M}\big) C_j + \sum_{1<l<j} \bar{S}_{lj} S_{lj} - 1 = 0 \end{aligned} \tag{3.2.42}$$

as well as:

$$\begin{aligned} &\bar{\lambda}_j S_{jk} + |\lambda_1|^{-2} C_k^\dagger C_1 C_1^\dagger C_j + C_k^\dagger C_j + \sum_{1<l<j} \bar{S}_{lk} S_{lj} = 0 \\ \Leftrightarrow\ &\bar{\lambda}_j S_{jk} + C_k^\dagger \big(C_1 C_1^\dagger + I_{l_M}\big) C_i + \sum_{1<l<j} \bar{S}_{lk} S_{lj} = 0 \end{aligned} \tag{3.2.43}$$

for $k > j$. We change variables again for $i = 2, \dots, N$

$$C_i^{(1)} = \sqrt{X_1} C_i \tag{3.2.44}$$

with $X_1 = |\lambda_1|^{-2} C_1 C_1^\dagger + I_{l_M}$ and Jacobian:

$$|\hat{J}_1| = \prod_{k=2}^{N} \det(X_1)^{-1}. \tag{3.2.45}$$

Applying Sylvester's determinant theorem then gives:

$$\det(X_1) = \det\big(|\lambda_1|^{-2}C_1^\dagger C_1 + 1\big). \tag{3.2.46}$$

Furthermore from equation (3.2.38):

$$\det(X_1) = \det\big(|\lambda_1|^{-2}(1-|\lambda_1|^2) + 1\big) = |\lambda_1|^{-2}. \tag{3.2.47}$$

Thus the Jacobian $|\hat{J}_1|$ cancels the Jacobian $|J_1|$ of the previous change of variables. As a result for $k = 2, \dots, N$, $k > j$:

$$\bar{\lambda}_j S_{jk} + (C_j^{(1)})^\dagger C_k^{(1)} + \sum_{1<l<j} \bar{S}_{lj} S_{lk} = 0 \tag{3.2.48}$$

$$|\lambda_j|^2 + (C_j^{(1)})^\dagger C_j^{(1)} + \sum_{1<l<j} |S_{lj}|^2 - 1 = 0. \tag{3.2.49}$$

Now the second row gives for $j = 3, \dots, N$:

$$\bar{\lambda}_2 S_{2k} + (C_2^{(1)})^\dagger C_k^{(1)} = 0. \tag{3.2.50}$$

Again we change variables for $k = 3, \dots, N$:

$$S_{2k}^{(1)} = \bar{\lambda}_2^{-1} S_{2k} \tag{3.2.51}$$

with Jacobian:

$$|J_2| = \prod_{k=3}^{N} |\lambda_2|^{-2}. \tag{3.2.52}$$

Note that $S_{2k}^{(1)} = -(C_2^{(1)})^\dagger C_k^{(1)}$. Using this relation for $j = 3, \dots, N$ yields:

$$\begin{aligned} &|\lambda_j|^2 + |\lambda_2|^{-2}(C_j^{(1)})^\dagger C_2^{(1)}(C_2^{(1)})^\dagger C_j^{(1)} + (C_j^{(1)})^\dagger C_j^{(1)} + \sum_{2<l<j} |S_{lj}|^2 - 1 = 0 \\ &\Leftrightarrow |\lambda_j|^2 + (C_j^{(1)})^\dagger\big(|\lambda_2|^{-2}C_2^{(1)}(C_2^{(1)})^\dagger + I_{l_M}\big)C_j^{(1)} + \sum_{2<l<j} |S_{lj}|^2 - 1 = 0 \end{aligned} \tag{3.2.53}$$

as well as:

$$\begin{aligned} &\bar{\lambda}_j S_{jk} + |\lambda_2|^{-2}(C_k^{(1)})^\dagger C_2^{(1)}(C_2^{(1)})^\dagger C_j^{(1)} + (C_k^{(1)})^\dagger C_j^{(1)} + \sum_{2<l<j} \bar{S}_{lk} S_{lj} = 0 \\ &\Leftrightarrow \bar{\lambda}_j S_{jk} + (C_k^{(1)})^\dagger\big(|\lambda_2|^{-2}C_2^{(1)}(C_2^{(1)})^\dagger + I_{l_M}\big)C_j^{(1)} + \sum_{2<l<j} \bar{S}_{lk} S_{lj} = 0 \end{aligned} \tag{3.2.54}$$

for $k > j$. We change variables again for $j = 3, \ldots, N$:

$$C_j^{(2)} = \sqrt{X_2} C_j^{(1)} \tag{3.2.55}$$

with $X_2 = |\lambda_2|^{-2} C_2^{(1)} (C_2^{(1)})^\dagger + I_{l_M}$ and Jacobian:

$$|\hat{J}_2| = \prod_{k=3}^{N} |\lambda_2|^2, \tag{3.2.56}$$

which cancels the previous Jacobian $|J_2|$. Consequently for $j = 3, \ldots, N$, $k > j$:

$$\bar{\lambda}_j S_{jk} + (C_j^{(2)})^\dagger C_k^{(2)} + \sum_{2<l<j} \bar{S}_{lj} S_{lk} = 0 \tag{3.2.57}$$

$$|\lambda_j|^2 + (C_j^{(2)})^\dagger C_J^{(2)} + \sum_{2<l<j} |S_{ki}|^2 - 1 = 0. \tag{3.2.58}$$

Repeating this procedure for all rows then yields:

$$I = \prod_{k=1}^{N} \int_{(C_k)} \delta\big(C_k^\dagger C_k + |\lambda_j|^2 - 1\big)(dC_k). \tag{3.2.59}$$

The last integral can be solved by a final change of variables:

$$D_k = C_k \sqrt{1 - |\lambda_k|^2} \tag{3.2.60}$$

with Jacobian $(1 - |\lambda_k|^2)^{l_M}$, whereas the delta function contributes a factor of $(1 - |\lambda_k|^2)^{-1}$. $\square$

Applying lemma 3.2.7 thus yields:

$$p(\lambda_1, \ldots, \lambda_N) = c_{\text{Stief}} \frac{\mathrm{Vol}(U(M))}{\mathrm{Vol}(U(L))\,\mathrm{Vol}(U(N))} \mathrm{Vol}(U[N]) \times \\ \left[\frac{\mathrm{Vol}(U(l_M))}{\mathrm{Vol}(U(l_M - 1))}\right]^N \prod_{j<k}^{N} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{N} |\lambda_j|^{2L} (1 - |\lambda_j|)^{l_M - 1}. \tag{3.2.61}$$

$\square$

Again setting the rectangularity parameter $L = 0$ we recover the eigenvalue jpdf of a square trunction of a random unitary matrix. The inducing procedure results in the additional factor $\prod_{j=1}^{N} |\lambda_j|^{2L}$. Again it should be noted that the eigenvalue jpdf is valid for $K \geq N + M$ as well as $K < N + M$. However the derivation of theorem 3.2.6 is only valid for integer values of L and $K - M - N$, as it relies on being able to apply the quadratization procedure from chapter 2 to a rectangular matrix of dimension $M \times N$.

3.2.3 The n-point correlation functions

Using the formalism of the method of orthogonal polynomials the $n-$point correlation function for both the complex induced spherical ensemble and the complex induced Jacobi ensemble can be easily derived. As already shown in theorem 3.1.9 the n-point correlation functions are given in the form:

$$R_n(\lambda_1, \dots, \lambda_n) = \det \Big(K_N(\lambda_k, \lambda_l) \Big)_{1 \le k,l \le n} \tag{3.2.62}$$

$$K_N(\lambda_k, \lambda_l) = w(\lambda_k) w(\lambda_l) \sum_{j=0}^{N-1} \frac{1}{r_j} p_j(\lambda_k) p_j(\bar{\lambda}_l). \tag{3.2.63}$$

All three induced ensembles are rotationally invariant. Thus as seen for the complex induced Gininbre ensemble the monomials can be used as the orthogonal polynomials inside the method of orthogonal polynomials. All that differs in the three cases is the weight function w as well as the normalization r_j. Further on let $w_{\text{IndSpherical},2}$ denote the weight function for the complex induced spherical ensemble with normalization $r_j^{\text{IndSpherical}}$, while $w_{\text{IndJacobi},2}$ denotes the weight function of the complex induced Jacobi ensemble with normalization $r_j^{\text{IndJacobi}}$. Then from the respective eigenvalue jpdf's:

$$w_{\text{IndSpherical},2}(\lambda) = \frac{\lambda^L}{(1+|\lambda|^2)^{\frac{n+L+1}{2}}} \tag{3.2.64}$$

$$w_{\text{IndJacobi},2}(\lambda) = \lambda^L (1-|\lambda|^2)^{\frac{l_M-1}{2}}. \tag{3.2.65}$$

Furthermore:

$$r_j^{\text{IndSpherical}} = \pi B(L+j+1, n-j) \tag{3.2.66}$$

$$r_j^{\text{IndJacobi}} = \pi B(L+j+1, l_M). \tag{3.2.67}$$

As a result,

Theorem 3.2.8. *The n-point correlation functions for the complex induced spher-*

ical ensemble are specified by the kernel:

$$K_N^{IndSpherical}(\lambda_k,\lambda_l) = \frac{1}{\pi}\big((1+|\lambda_k|^2)(1+|\lambda_l|^2)\big)^{-\frac{n+L+1}{2}} \times \sum_{j=0}^{N-1}\frac{\Gamma(n+L+1)}{\Gamma(j+L+1)\Gamma(n-j)}\big(\lambda_k\bar{\lambda}_l\big)^{j+L}, \tag{3.2.68}$$

with the mean eigenvalue density:

$$R_1^{IndSpherical}(\lambda) = \frac{1}{\pi}(1+|\lambda|^2)^{-n-L-1}\sum_{j=0}^{N-1}\frac{\Gamma(n+L+1)}{\Gamma(j+L+1)\Gamma(n-j)}|\lambda|^{2(L+j)}. \tag{3.2.69}$$

Meanwhile,

Theorem 3.2.9. *The n-point correlation functions for the complex induced Jacobi ensemble are specified by the kernel:*

$$K_N^{IndJacobi}(\lambda_k,\lambda_l) = \frac{1}{\pi}\big((1-|\lambda_k|^2)(1-|\lambda_l|^2)\big)^{\frac{l_M-1}{2}} \times \sum_{j=0}^{N-1}\frac{\Gamma(l_N+j+1)}{\Gamma(j+L+1)\Gamma(l_M)}\big(\lambda_k\bar{\lambda}_l\big)^{j+L}, \tag{3.2.70}$$

with the mean eigenvalue density:

$$R_1^{IndJacobi}(\lambda) = \frac{1}{\pi}(1-|\lambda|^2)^{l_M-1}\sum_{j=0}^{N-1}\frac{\Gamma(l_N+j+1)}{\Gamma(j+L+1)\Gamma(l_M)}|\lambda|^{2(L+j)}. \tag{3.2.71}$$

Again for the sake of the asymptotic analysis it useful to derive integral representations for the eigenvalue statistics. Remarkably it is possible to express the mean eigenvalue densities of the complex induced Jacobi and the complex induced spherical ensemble using a difference of incomplete beta functions.

Lemma 3.2.10.

$$\begin{aligned} S_{IndSpherical}(z) :=& \sum_{j=0}^{N-1}\frac{\Gamma(n+L+1)}{\Gamma(j+L+1)\Gamma(n-j)}z^j \qquad (3.2.72)\\ =&(n+L)\frac{(1+z)^{n+L-1}}{z^L}\big[J_z(L,n)-J_z(M,n-N)\big] \\ S_{IndJacobi}(z) :=& \sum_{j=0}^{N-1}\frac{\Gamma(l_N+j+1)}{\Gamma(j+L+1)\Gamma(l_M)}z^j \qquad (3.2.73)\\ =&\frac{l_M}{z^L(1-z)^{l_M+1}}\big[I_z(L,l_M+1)-I_z(M,l_M+1)\big] \end{aligned}$$

with

$$J_z(a,b) = \frac{1}{B(a,b)} \int_0^z \frac{t^{a-1}}{(1+t)^{a+b}} = I_{\frac{z}{1+z}}(a,b).$$

Proof. Straightforward working, as seen in the proof of lemma 3.1.10, establishes that the sums satisfy the following first order differential equation:

$$S_I'(z) + h_I(z)S_I(z) = c_I(z), \quad \text{for } I \in \{\text{IndSpherical}, \text{IndJacobi}\}$$

where,

$$\begin{aligned}
h_{\text{IndSpherical}}(z) &= \frac{1}{1+z}\Big(\frac{L}{z} - n + 1\Big) \\
h_{\text{IndJacobi}}(z) &= \frac{1}{1-z}\Big(\frac{L}{z} - l_N - 1\Big) \\
c_{\text{IndSpherical}}(z) &= \frac{L}{z(1+z)} \frac{\Gamma(n+L+1)}{\Gamma(L+1)\Gamma(n)} - \frac{z^{N-1}}{1+z} \frac{\Gamma(n+L+1)}{\Gamma(M)\Gamma(n-N)} \\
c_{\text{IndJacobi}}(z) &= \frac{L}{z(1-z)} \frac{\Gamma(l_N+1)}{\Gamma(L+1)\Gamma(l_M)} - K\frac{z^{N-1}}{1-z} \frac{\Gamma(K)}{\Gamma(M)\Gamma(l_M)}.
\end{aligned}$$

The differential equation can be solved using the method of integrating factors, choosing $I_I(z)$ such that:

$$I_I'(z) = h_I(z)I_I(z) \tag{3.2.74}$$

implying:

$$\Big(I_I(z)S_I(z)\Big)' = c_I(z)I_I(z). \tag{3.2.75}$$

Solving (3.2.74) yields:

$$I_{\text{IndSpherical}}(z) = \frac{z^L}{(1+z)^{n+L-1}} \tag{3.2.76}$$

$$I_{\text{IndJacobi}}(z) = z^L(1-z)^{l_M+1}. \tag{3.2.77}$$

Consequently:

$$\begin{aligned}
S_{\text{IndSpherical}}(z) =& \frac{(1+z)^{n+L+1}}{z^L}\Big(\frac{L}{B(L+1,n)} \int_0^z \frac{t^{L-1}}{(1+t)^{n+L}} dt \\
&- \frac{n-N}{B(M,n-N+1)} \int_0^z \frac{t^{M-1}}{(1+t)^{n+L}} dt\Big)
\end{aligned} \tag{3.2.78}$$

$$\begin{aligned}
S_{\text{IndJacobi}}(z) =& \frac{1}{z^L(1-z)^{l_M+1}}\Big(\frac{L}{B(L,l_M)} \int_0^z t^{L-1}(1-t)^{l_M} dt \\
&- \frac{K}{B(M,l_M)} \int_0^z t^{M-1}(1-t)^{l_M} dt\Big).
\end{aligned} \tag{3.2.79}$$

Using the recurrences:

$$B(x+1,y) = \frac{x}{x+y}B(x,y), \qquad B(x,y+1) = \frac{y}{x+y}B(x,y),$$

then gives the desired result. □

3.2.4 The induced spherical ensemble: Asymptotic analysis

In the following section the asymptotic behavior of the eigenvalue statistics of the complex induced spherical ensemble in the limit of large matrix dimensions is outlined. There are four distinct asymptotic regimes in the following asymptotic analysis, depending on the behavior of the rectangularity parameter L and a second rectangularity parameter coming from the Wishart matrices used to generate the complex induced spherical ensemble. This parameter, $n-N$ is hereafter referred to as the spherical component. The whole asymptotic analysis can be undertaken by exploiting the properties of the beta function as outlined and proven in section A.2.

The main distinction between the different asymptotic regimes is the support of the limiting eigenvalue density. However after an inverse stereographical projection to the unit sphere the eigenvalues are either uniformly distributed on a so-called spherical annulus or on the entire sphere. A spherical annulus is a surface on the sphere, obtained by drawing two parallel circles on the sphere, as shown in figure 3.2 (a). In all four regimes, in the bulk of the eigenvalue support the limiting correlation kernel shows universal behavior. More precisely the correlation kernels of the complex induced Ginibre ensemble (from theorem 3.1.13 and equation (3.1.71)) are found in the respective asymptotic regimes.

Figure 3.2 shows the eigenvalue distribution of the complex induced spherical ensemble in the four asymptotic regimes, while figure 3.3 shows the eigenvalue distribution of the complex induced spherical ensemble after an inverse stereographical projection to the unit sphere. Starting point of the asymptotic analysis is the mean eigenvalue density. Using the integral representation derived in lemma 3.2.10 the mean eigenvalue density of the complex induced spherical ensemble is given by:

$$R_1^{\mathrm{IndSpherical}}(z) = \frac{1}{\pi}\frac{n+L}{\left(1+|z|^2\right)^2}\left[I_{\frac{|z|^2}{1+|z|^2}}(L,n) - I_{\frac{|z|^2}{1+|z|^2}}(M,n-N)\right]. \qquad (3.2.80)$$

We introduce the scaled mean eigenvalue density:

$$\rho_N^{\text{IndSpherical}}(z) = \frac{1}{n+L} R_1^{\text{IndSpherical}}(z). \tag{3.2.81}$$

The correlation kernel for the complex induced spherical ensemble takes the form:

$$K_N^{\text{IndSpherical}}(z_k, z_l) = \frac{n+L}{\pi} \frac{(1+z_k\bar{z}_l)^{n+L-1}}{[(1+|z_k|^2)(1+|z_l|^2)]^{\frac{n+L+1}{2}}} \times \\ \left[I_{\frac{z_k\bar{z}_l}{1+z_k\bar{z}_l}}\big(L,n\big) - I_{\frac{z_k\bar{z}_l}{1+z_k\bar{z}_l}}\big(M,n-N\big)\right]. \tag{3.2.82}$$

Regime 1: Strong rectangularity and strong spherical component

In this asymptotic regime the rectangularity parameter is scaled proportional to matrix size $L = N\alpha$, while the spherical component is also scaled proportional to matrix size $n - N = N\beta$. In addition set $\frac{L}{n} := \mu_1$ and $\frac{M}{n-N} =: \mu_2$.

In the limit of large matrix dimensions the mean eigenvalue density is supported on an annulus of width $\sqrt{\mu_2} - \sqrt{\mu_1}$. Thus the density has two cut-offs, the inner edge with radius $r^{\text{in}} = \sqrt{\mu_1}$ and the outer edge with radius $r^{\text{out}} = \sqrt{\mu_2}$. Close to the edges of the eigenvalue support the density exhibits universal behavior of the Feinberg-Zee type for $\beta = 2$ as predicted in [Bog10]. Furthermore in the limit of large matrix dimensions the n-point correlation functions after unfolding likewise exhibit universal behavior on the support of the eigenvalue density. This means that after scaling the reference point with the mean eigenvalue density we recover the eigenvalue correlations of the complex Ginibre ensemble from theorem 3.1.13. More precisely,

Theorem 3.2.11. *In the regime of strong rectangularity and strong spherical component, $L = N\alpha$ and $n - N = N\beta$ in the limit of large matrix dimension N the mean eigenvalue density is given by:*

$$\lim_{N\to\infty} \rho_N^{IndSpherical}(z) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\left[\Theta(|z|-\sqrt{\mu_1}) - \Theta(|z|-\sqrt{\mu_2})\right] =: \rho^{IndSpherical}(z). \tag{3.2.83}$$

At the edges of $z^{in} = \left(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\right)\mathrm{e}^{i\phi}$ and $z^{out} = \left(\sqrt{\mu_2} + \frac{\xi}{\sqrt{n+L}}\right)\mathrm{e}^{i\phi}$ of the *eigenvalue support:*

$$\lim_{N\to\infty} \rho_N^{IndSpherical}\big(z^{in}\big) = \pi\rho^{IndSpherical}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\,\mathrm{erfc}\Big(\sqrt{2\pi\rho^{IndSpherical}\big(\sqrt{\mu_1}\big)}\xi\Big)$$
$$= \frac{1}{2\pi}\frac{1}{(1+\mu_1)^2}\,\mathrm{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big) \tag{3.2.84}$$
$$\lim_{N\to\infty} \rho_N^{IndSpherical}\big(z^{out}\big) = \pi\rho^{IndSpherical}\big(\sqrt{\mu_2}\big)\frac{1}{2\pi}\,\mathrm{erfc}\Big(\sqrt{2\pi\rho^{IndSpherical}\big(\sqrt{\mu_2}\big)}\xi\Big)$$
$$= \frac{1}{2\pi}\frac{1}{(1+\mu_2)^2}\,\mathrm{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\Big). \tag{3.2.85}$$

Furthermore for the correlation kernel at the bulk set $z_k = u + \frac{s_k}{\sqrt{n+L}}$*,* $k = 1, \dots, N$*, where* $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$*. Then:*

$$\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) = \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)}$$
$$= \pi\rho^{IndSpherical}(u)\frac{1}{\pi}\,\mathrm{e}^{-\sqrt{\pi\rho^{IndSpherical}(u)}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)}. \tag{3.2.86}$$

Proof. Note that:

$$\frac{|z|^2}{1+|z|^2} < \frac{\frac{\alpha}{\beta+1}}{1+\frac{\alpha}{\beta+1}} \quad \Leftrightarrow \quad |z|^2 < \frac{\alpha}{\beta+1} := \mu_1. \tag{3.2.87}$$

Hence applying theorem A.2.1 yields:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}\big(N\alpha, N(\beta+1)\big) = \Theta\big(|z| - \sqrt{\mu_1}\big). \tag{3.2.88}$$

Similarly another application of theorem A.2.1 gives:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}\big(N(\alpha+1), N\beta\big) = \Theta\big(|z| - \sqrt{\mu_2}\big), \tag{3.2.89}$$

which proves the asymptotic form of the mean eigenvalue density. Now around the edges of the eigenvalue support set: $z^{\mathrm{in}} = \big(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\big)\,\mathrm{e}^{i\phi}$ at the inner edge and $z^{\mathrm{out}} = \big(\sqrt{\mu_2} + \frac{\xi}{\sqrt{n+L}}\big)\,\mathrm{e}^{i\phi}$ the outer edge . Utilizing the asymptotic properties of the beta function from theorem A.2.5, the edge relation follows immediately. Next we study the correlation functions, first at the bulk with scaling $z_k = u + \frac{s_k}{\sqrt{n+L}}$ 1, $k = 1, \ldots, N$, where $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$. Note that:

$$
\begin{aligned}
&\frac{\left[1+\left(u+\frac{s_k}{\sqrt{n+L}}\right)\left(\bar{u}+\frac{\bar{s}_l}{\sqrt{n+L}}\right)\right]^{n+L-1}}{\left(1+\left(u+\frac{s_k}{\sqrt{n+L}}\right)\left(\bar{u}+\frac{\bar{s}_k}{\sqrt{n+L}}\right)\right)^{\frac{n+L+1}{2}}\left(1+\left(u+\frac{s_l}{\sqrt{n+L}}\right)\left(\bar{u}+\frac{\bar{s}_l}{\sqrt{n+L}}\right)\right)^{\frac{n+L+1}{2}}}\\
=&\frac{(1+|u|^2)^{n+L-1}\left(1+\frac{1}{\sqrt{n+L}(1+|u|^2)}(s_k\bar{u}+\bar{s}_l u)+\frac{1}{(n+L)(1+|u|^2)}s_k\bar{s}_l\right)^{n+L-1}}{(1+|u|^2)^{n+L+1}\left(1+\frac{1}{\sqrt{n+L}(1+|u|^2)}(s_k\bar{u}+\bar{s}_k u)+\frac{1}{(n+L)(1+|u|^2)}|s_k|^2\right)^{\frac{n+L-1}{2}}}\times\\
&\left(1+\frac{1}{\sqrt{n+L}(1+|u|^2)}(s_l\bar{u}+\bar{s}_l u)+\frac{1}{(n+L)(1+|u|^2)}|s_l|^2\right)^{-\frac{n+L-1}{2}}\\
&\sim\frac{1}{(1+|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\,. \qquad (3.2.90)
\end{aligned}
$$

Together with the asymptotic properties of the incomplete beta function from section A.2, we obtain:

$$
K_N^{\text{IndSpherical}}(z_k, z_l) \sim \frac{n+L}{\pi}\frac{1}{(1+|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\,. \qquad (3.2.91)
$$

□

Strong rectangularity and weak spherical component

In the asymptotic regime of strong rectangularity and weak spherical component the rectangularity parameter is again scaled proportional to matrix size $L = N\alpha$, while the spherical component $n - N = O(1)$ is kept fixed. In addition we set $\frac{N\alpha}{n-1} = \frac{N\alpha}{N+n-N-1} \sim \alpha := \mu_1$.

As a result the mean eigenvalue density is supported on the whole complex plane except on a disk around the origin with radius $r^{\text{in}} = \sqrt{\mu_1}$. Consequently the eigenvalue density possesses only one, circular edge, at which the density falls to zero at Gaussian rate and the mean density exhibits universal behavior of the Feinberg-Zee type for $\beta = 2$. Furthermore the correlation kernel is again universal on the support of the eigenvalue density, meaning that after scaling our reference point with the mean eigenvalue density, we recover the correlation kernel of the complex Ginibre ensemble from theorem 3.1.13. More precisely,

Theorem 3.2.12. *In the regime of strong rectangularity and weak spherical component, $L = N\alpha$ and $n - N = O(1)$ in the limit of large matrix dimension N the* *mean eigenvalue density is given by:*

$$\lim_{N\to\infty} \rho_N^{IndSpherical}(z) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\,\Theta\big(|z|-\sqrt{\mu_1}\,\big) =: \rho^{IndSpherical}(z). \qquad (3.2.92)$$

At the edge of $z^{in} = \big(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\,\big)\,\mathrm{e}^{i\phi}$ *of the eigenvalue support:*

$$\begin{aligned}\lim_{N\to\infty} \rho_N^{IndSpherical}\big(z^{in}\big) =& \pi\rho^{IndSpherical}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\,\mathrm{erfc}\,\Big(\sqrt{2\pi\rho^{IndSpherical}\big(\sqrt{\mu_1}\big)}\xi\Big)\\ =& \frac{1}{2\pi}\frac{1}{(1+\mu_1)^2}\,\mathrm{erfc}\,\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big).\end{aligned} \qquad (3.2.93)$$

Furthermore for the correlation kernel at the bulk set $z_k = u+\frac{s_k}{\sqrt{n+L}}$, $k = 1,\ldots,N$, *where* $\sqrt{\mu_1} < |u|$. *Then:*

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) =& \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)}\\ =& \pi\rho^{IndSpherical}(u)\frac{1}{\pi}\,\mathrm{e}^{-\sqrt{\pi\rho^{IndSpherical}(u)}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)}\,.\end{aligned} \qquad (3.2.94)$$

Proof. As in the proof of theorem 3.2.12 applying theorem A.2.1 gives:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(L,n) = \Theta\big(|z|-\sqrt{\mu_1}\,\big). \qquad (3.2.95)$$

In addition using theorem A.2.2 yields:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(M, n-N) = 0, \qquad (3.2.96)$$

which gives the asymptotic form of the mean eigenvalue density at the bulk. At the inner edge $z^{\mathrm{in}} = \big(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\,\big)\,\mathrm{e}^{i\phi}$, we can apply A.2.5 and obtain the limiting expression for the eigenvalue density at the circular edge. Together with the proof of 3.2.11 the derivation of the correlation kernel asymptotics is a straightforward application of theorem A.2.6. □

Almost square and strong spherical component

In the regime of almost square matrices with strong spherical component the rectangularity parameter $L = O(1)$ is kept fixed, while the spherical component grows proportionally to matrix size $n-N = N\beta$. Furthermore we set $\frac{N+L-1}{N\beta} \sim \frac{1}{\beta} := \mu_2$.

As a result the mean eigenvalue density is supported on a disk around the origin with radius $r^{\mathrm{out}} = \sqrt{\mu_2}$. Consequently the density possesses one outer edge at which the density falls to zero at Gaussian rate and the mean density exhibits universal behavior. Furthermore the correlation kernel is again universal on the support of the eigenvalue density, while at the origin the correlation kernel of the complex induced Ginibre ensemble is recaptured. More precisely,

Theorem 3.2.13. *In the regime of almost square matrices with strong spherical component, $L = O(1)$ and $n - N = N\beta$ in the limit of large matrix dimension N the mean eigenvalue density is given by:*

$$\lim_{N\to\infty} \rho_N^{IndSpherical}(z) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\,\Theta\big(\sqrt{\mu_2} - |z|\,\big) = \rho^{IndSpherical}(z). \qquad (3.2.97)$$

At the edge $z^{out} = \big(\sqrt{\mu_2} + \frac{\xi}{\sqrt{n+L}}\,\big)\,\mathrm{e}^{i\phi}$ of the eigenvalue support:

$$\lim_{N\to\infty} \rho_N^{IndSpherical}\big(z^{out}\big) = \frac{1}{2\pi}\frac{1}{(1+\mu_2)^2}\,\mathrm{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\Big). \qquad (3.2.98)$$

Furthermore for the correlation kernel at the bulk set $z_k = u + \frac{s_k}{\sqrt{n+L}}$, $k = 1, \ldots, N$, where $0 < |u| < \sqrt{\mu_2} < \infty$. Then:

$$\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) = \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)} : \qquad (3.2.99)$$

At the origin $u = 0$, $z_k = \frac{s_k}{\sqrt{n+L}}$ we obtain:

$$\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) = \frac{1}{\pi}\,\mathrm{e}^{\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2 - s_k\bar{s}_l}\,\frac{\gamma(s_k\bar{s}_l, L)}{\Gamma(L)}. \qquad (3.2.100)$$

Proof. Again using theorem A.2.2 gives:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(L, n) = 1, \qquad (3.2.101)$$

while applying theorem A.2.1 yields:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(L+N-1, n-N) = \Theta\Big(|z| - \sqrt{\mu_2}\Big). \qquad (3.2.102)$$

The statements concerning the mean eigenvalue density at the edge as well as the correlation functions in the bulk are again straightforward applications of theorem A.2.5 and theorem A.2.6. For the correlation kernel at the origin with scaling $z_k = \frac{s_k}{\sqrt{n+L}}$ we note:

$$K_N^{\text{IndSpherical}}(z_k, z_l) = \frac{n+L}{\pi} \frac{\left(1 + \frac{s_k \bar{s}_l}{n+L}\right)^{n+L-1}}{\left[\left(1 + \frac{|s_k|^2}{n+L}\right)\left(1 + \frac{|s_l|^2}{n+L}\right)\right]^{\frac{n+L+1}{2}}} \times \left[J_{\frac{s_k \bar{s}_l}{n+L}}(L, n+L) - J_{\frac{s_k \bar{s}_l}{n+L}}(M, n+L)\right]. \tag{3.2.103}$$

It follows that:

$$\frac{\left(1 + \frac{s_k \bar{s}_l}{n+L}\right)^{n+L-1}}{\left[\left(1 + \frac{|s_k|^2}{n+L}\right)\left(1 + \frac{|s_l|^2}{n+L}\right)\right]^{\frac{n+L+1}{2}}} \sim \mathrm{e}^{-\frac{1}{2}|s_k|^2 - \frac{1}{2}|s_l|^2 + s_k \bar{s}_l}. \tag{3.2.104}$$

Furthermore:

$$\begin{aligned} J_{\frac{s_k \bar{s}_l}{n+L}}(L, n+L) =& \frac{1}{B(L, n+L)} \int_0^{\frac{s_k \bar{s}_l}{n+L}} \frac{t^{L-1}}{(1+t)^{n+L}} dt \\ \sim& \frac{1}{\Gamma(L)} (n+L)^L \frac{1}{(n+L)^L} \int_0^{s_k \bar{s}_l} \frac{t^{L-1}}{(1 + \frac{t}{n+L})^{n+L}} dt \\ \sim& \frac{1}{\Gamma(L)} \int_0^{s_k \bar{s}_l} t^{L-1} \mathrm{e}^{-t} \, dt. \end{aligned} \tag{3.2.105}$$

Similarly:

$$\begin{aligned} J_{\frac{s_k \bar{s}_l}{n+L}}(M, n+L) =& \frac{1}{B(M, n+L)} \int_0^{\frac{s_k \bar{s}_l}{n+L}} \frac{t^{M-1}}{(1+t)^{n+L}} dt \\ \sim& \frac{1}{B(M, n+L)} \frac{1}{(n+L)^L} \int_0^{s_k \bar{s}_l} t^{M-1} \mathrm{e}^{-t} \, dt. \end{aligned} \tag{3.2.106}$$

□

Almost square and weak spherical component

In the regime of almost square matrices with weak spherical component both parameters are kept fixed $L, n - N = O(1)$. As a result in the limit of large matrix dimensions the mean eigenvalue density is supported on the whole complex plane and the eigenvalues are standard Chauchy distributed. Furthermore the correlation kernel in the bulk is again universal, while at the origin the correlation kernel of the induced Ginibre ensemble at the origin from equation (3.1.71) is recaptured.

Theorem 3.2.14. *In the regime of almost square matrices with weak spherical component, $L = O(1)$ and $n - N = O(1)$ in the limit of large matrix dimension*

N the mean eigenvalue density is given by:

$$\lim_{N\to\infty} \rho_N^{IndSpherical}(z) = \frac{1}{\pi}\,\frac{1}{(1+|z|^2)^2} := \rho^{IndSpherical}(z). \tag{3.2.107}$$

Furthermore for the correlation kernel at the bulk set $z_k = u + \frac{s_k}{\sqrt{n+L}}$, $k = 1, \ldots, N$. Then:

$$\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) = \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}\, \mathrm{e}^{-\frac{1}{1+|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}. \tag{3.2.108}$$

At the origin $u = 0$, $z_k = \frac{s_k}{\sqrt{n+L}}$ we obtain:

$$\lim_{N\to\infty} \frac{1}{n+L} K_N^{IndSpherical}(z_k, z_l) = \frac{1}{\pi}\, \mathrm{e}^{\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l} \,.\frac{\gamma(s_k\bar{s}_l, L)}{\Gamma(L)}. \tag{3.2.109}$$

3.2.5 The induced Jacobi ensemble: Asymptotic analysis

In the following section the asymptotic behavior of the eigenvalue statistics of the complex induced Jacobi ensemble is analyzed. Again we can distinguish four asymptotic regimes, which depend on the rectangularity parameter L and the parameter l_M. The parameter l_M, defined in equation (3.2.22), controls how many rows are deleted from the unitary matrix $Q \in \mathbb{C}^{K\times K}$, which is used in the generation of induced Jacobi matrices. In the following the regime of strong non-unitarity refers to truncations of unitary matrices by deleting a finite number of rows and columns. Similarly the term strong non-unitarity shall refer to truncations of unitary matrices, which are obtained by deleting a number of rows and columns, that is proportional to matrix size. The induced Jacobi ensemble possesses the most interesting asymptotic behavior of the induced family of random matrix ensembles. In the regime of strong rectangularity and partially weak non-unitarity, which will be defined below, a new limiting correlation kernel is discovered. This is one of the main results of this work. Furthermore in the limit of strong non-unitarity the correlation kernels of the complex induced Ginibre ensemble is found, while in the regime of almost square matrices with weak non-unitarity the correlation functions of square trunctions of random unitary matrices in the regime of weak non-unitarity are recovered [SŻ00, KS09]. An overview of the different asymptotic regimes and the prevalent limiting correlation kernels is given in section 3.3.2.

Figure 3.4 shows the eigenvalue distribution of the complex induced Jacobi ensemble in the four asymptotic regimes.

Again the asymptotic analysis is undertaken by exploiting the beta function asymptotics outlined in appendix A. Starting point is the integral representation of mean density of complex eigenvalues:

$$R_1^{\mathrm{IndJacobi}}(z) = \frac{1}{\pi}\frac{l_M}{\left(1-|z|^2\right)^2}\cdot\left[I_{|z|^2}(L, l_M+1) - I_{|z|^2}(M, l_M+1)\right]. \qquad (3.2.110)$$

Again we introduce the scaled mean eigenvalue density:

$$\rho_N^{\mathrm{IndJacobi}}(z) = \frac{1}{n+L}R_1^{\mathrm{IndJacobi}}(z). \qquad (3.2.111)$$

The correlation kernel takes the form:

$$\begin{aligned} K_N^{\mathrm{IndJacobi}}(z_k, z_l) = & \frac{l_M}{\pi}\,\frac{\left[(1-|z_k|^2)(1-|z_l|^2)\right]^{\frac{l_M-1}{2}}}{(1-z_k\bar{z}_l)^{l_M+1}} \times \\ & \Big[I_{z_k\bar{z}_l}\big(L, l_M+1\big) - I_{z_k\bar{z}_l}\big(M, l_M+1\big)\Big]. \end{aligned} \qquad (3.2.112)$$

Strong rectangularity and strong non-unitarity

In the regime of strong rectangularity and strong non-unitarity the rectangularity parameter L grows proportionally with matrix size: $L = N\alpha$, while the size of the induced matrix grows proportionally with the size of the unitary matrix: $K = kN$. This implies the following relations: $l_N = (k-1)N$, $M = (\alpha+1)N$ and $l_M = (k-\alpha-1)N$. The number of deleted rows and columns grows proportionally with matrix size. In addition set $\mu_1 := \frac{L}{l_N}$ and $\mu_2 := \frac{M}{K}$

As a consequence in the limit of large matrix dimensions the eigenvalues are distributed on an annulus of width $\sqrt{\mu_2} - \sqrt{\mu_1}$. Thus the density has two cut-offs, the inner edge with radius $r^{\mathrm{in}} = \sqrt{\mu_1}$ and the outer edge with radius $r^{\mathrm{out}} = \sqrt{\mu_2}$. Close to the edges of the eigenvalue support the density exhibits universal behavior as predicted in Eqs. (42)–(43) in [Bog10] of the Feinberg-Zee type for $\beta = 2$. Furthermore in the limit of large matrix dimensions the n-point correlation functions after unfolding likewise exhibit universal behavior. Specifically after scaling the reference point with the limiting mean eigenvalue density the correlation kernel of the complex Ginibre ensemble from theorem 3.1.13 is found. More precisely,

Theorem 3.2.15. *In the regime of strong rectangularity and strong non-unitarity, $L = N\alpha$ and $l_M = (k-\alpha-1)N$ in the limit of large matrix dimension the mean* *eigenvalue density is given by:*

$$\lim_{N\to\infty}\rho_N^{IndJacobi}(z)=\frac{1}{\pi}\frac{1}{(1-|z|^2)^2}\big[\Theta\big(|z|-\sqrt{\mu_1}\,\big)-\Theta\big(|z|-\sqrt{\mu_2}\,\big)\big]:=\rho^{IndJacobi}(z). \tag{3.2.113}$$

At the edges $z^{in}=\big(\sqrt{\mu_1}-\frac{\xi}{\sqrt{l_M}}\,\big)\,\mathrm{e}^{i\phi}$ *and* $z^{out}=\big(\sqrt{\mu_2}+\frac{\xi}{\sqrt{l_M}}\,\big)\,\mathrm{e}^{i\phi}$ *of the eigenvalue support:*

$$\begin{aligned}\lim_{N\to\infty}\rho_N^{IndJacobi}\big(z^{in}\big)=&\pi\rho^{IndJacobi}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\operatorname{erfc}\Big(\sqrt{2\pi\rho^{IndJacobi}\big(\sqrt{\mu_1}\big)}\xi\Big)\\=&\frac{1}{2\pi}\frac{1}{(1-\mu_1)^2}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1-\mu_1}}\xi\Big)\end{aligned}\tag{3.2.114}$$

$$\begin{aligned}\lim_{N\to\infty}\rho_N^{IndJacobi}\big(z^{out}\big)=&\pi\rho^{IndJacobi}\big(\sqrt{\mu_2}\big)\frac{1}{2\pi}\operatorname{erfc}\Big(\sqrt{2\pi\rho^{IndJacobi}\big(\sqrt{\mu_2}\big)}\xi\Big)\\=&\frac{1}{2\pi}\frac{1}{(1-\mu_2)^2}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\Big).\end{aligned}\tag{3.2.115}$$

Furthermore for the correlation kernel at the bulk set $z_k=u+\frac{s_k}{\sqrt{n+L}}$, $k=1,\dots,N$, *where* $u,s_k\in\mathbb{C}$ $\sqrt{\mu_1}<|u|<\sqrt{\mu_2}$. *Then:*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{l_M}K_N^{IndJacobi}(z_k,z_l)=&\frac{1}{\pi}\frac{1}{(1-|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1-|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\\=&\pi\rho^{IndJacobi}(u)\frac{1}{\pi}\,\mathrm{e}^{-\sqrt{\pi\rho^{IndJacobi}(u)}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\,.\end{aligned}\tag{3.2.116}$$

Proof. It follows from theorem A.2.1, that:

$$\lim_{N\to\infty}I_{|z|^2}\big(L,l_M+1\big)=\Theta\big(\,|z|-\sqrt{\mu_1}\,\big), \tag{3.2.117}$$

as well as:

$$\lim_{N\to\infty}I_{|\lambda|^2}\big(M,l_M+1\big)=\Theta\big(\,|z|-\sqrt{\mu_2}\,\big). \tag{3.2.118}$$

Consequently the mean eigenvalue density is supported on a ring about the origin with radii $r_{\text{in}}=\sqrt{\mu_1}$ and $r_{\text{out}}=\sqrt{\mu_2}$. The edge profile at the inner and outer edge can be show by applying theorem A.2.1 and theorem A.2.4 to the respective incomplete beta functions.

Next we study the correlation kernel in the bulk with scaling $z_k=u+\frac{s_k}{\sqrt{l_M}}$

for $k =, \ldots, N$ where $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$. Note first that:

$$
\begin{aligned}
&\frac{\left(1-\left(u+\frac{s_k}{\sqrt{l_M}}\right)\left(\bar{u}+\frac{\bar{s}_k}{\sqrt{l_M}}\right)\right)^{\frac{l_M-1}{2}}\left(1-\left(u+\frac{s_l}{\sqrt{l_M}}\right)\left(\bar{u}+\frac{\bar{s}_l}{\sqrt{l_M}}\right)\right)^{\frac{l_M-1}{2}}}{\left[1-\left(u+\frac{s_k}{\sqrt{l_M}}\right)\left(\bar{u}+\frac{\bar{s}_l}{\sqrt{l_M}}\right)\right]^{l_M+1}} \\
=&\frac{(1-|u|^2)^{l_M-1}\left(1-\frac{1}{\sqrt{N}(1-|u|^2)}(s_k\bar{u}+\bar{s_k}u)-\frac{1}{N(1-|u|^2)}|s_k|^2\right)^{\frac{l_M-1}{2}}}{(1-|u|^2)^{l_M+1}\left(1-\frac{1}{\sqrt{l_M}(1-|u|^2)}(s_k\bar{u}+\bar{s}_l u)-\frac{1}{N(1-|u|^2)}s_k\bar{s}_l\right)^{l_M+1}}\times \\
&\left(1-\frac{1}{\sqrt{l_M}(1-|u|^2)}(s_l\bar{u}+\bar{s}_l u)-\frac{1}{l_M(1-|u|^2)}|z_l|^2\right)^{\frac{l_M-1}{2}} \\
&\sim\frac{1}{(1-|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1-|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\,.
\end{aligned}
\tag{3.2.119}
$$

Together with the asymptotic properties of the incomplete beta function from theorem A.2.6, we obtain:

$$
K_N^{\mathrm{IndJacobi}}(z_k,z_l)\sim\frac{l_M}{\pi}\frac{1}{(1-|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1-|u|^2}\left(\frac{1}{2}|s_k|^2+\frac{1}{2}|s_l|^2-s_k\bar{s}_l\right)}\,. \tag{3.2.120}
$$

□

Strong rectangularity and partially weak non-unitarity

In the regime of strong rectangularity and partially weak on-unitarity the rectangularity parameter is proportionally to matrix size: $L = N\alpha$, while the parameter controlling the unitarity of the induced Jacobi matrix l_M is kept fixed. Note that strong rectangularity implies: $l_N = K - N = K - \frac{1}{1+\alpha}(K - l_M) = \frac{\alpha}{1+\alpha}K - \frac{l_M}{1+\alpha}$. The number of deleted rows is fixed, while the number of deleted columns grows proportionally with matrix size.

In this regime the eigenvalues of A lie close to the unit circle. More importantly a new type of correlation kernel emerges, extending the number of known universality classes for non-hermitian random matrix ensembles for $\beta = 2$. More precisely,

Theorem 3.2.16. *Set* $z = (1-\frac{y}{N})\,\mathrm{e}^{i\phi}$. *Then in the regime of strong rectangularity and partially weak non-unitarity in the limit of large matrix dimensions the mean* *eigenvalue density is given by:*

$$\lim_{N\to\infty}\frac{1}{N^2}R_1^{IndJacobi}\big((1-\frac{y}{N})\,\mathrm{e}^{i\phi}\big)=\frac{1}{\pi}\frac{\Gamma(2y(\alpha+1),l_M+1)-\Gamma(2y\alpha,l_M+1)}{\Gamma(l_M)}. \tag{3.2.121}$$

For the correlation kernel at the bulk we scale $z_k=(1-\frac{y_k}{N})\,\mathrm{e}^{i(\phi_0+\frac{\phi_k}{N})}$, then:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^2}K_N^{IndJacobi}(z_k,z_l)\\ =&\frac{1}{\pi}\frac{\big(2\sqrt{y_ky_l}\big)^{l_M-1}}{\Gamma(l_M)}\left[\int_0^{\alpha+1}s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds-\int_0^{\alpha}s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds\right].\end{aligned} \tag{3.2.122}$$

Proof. It follows that: $|z|=1-\frac{y}{N}$ and $|z|^2=1-\frac{2y}{N}+\frac{y^2}{N^2}\sim 1-\frac{2y}{N}$. Then using theorem A.2.2 yields:

$$\begin{aligned}\lim_{N\to\infty}I_{1-\frac{2y}{N}+\frac{y^2}{N^2}}(N\alpha,l_M+1)&=1-\frac{(2y\alpha)^{l_M+1}}{\Gamma(l_M+1)}\int_0^1 s^{l_m}e^{-2y\alpha s}ds\\ &=1-\frac{(1}{\Gamma(l_M+1)}\int_0^{2y\alpha}s^{l_m}e^{-s}ds\\ &=\frac{\Gamma(2y\alpha,l_M+1}{\Gamma(l_M+1)},\end{aligned} \tag{3.2.123}$$

as well as:

$$\lim_{N\to\infty}I_{1-\frac{2y}{N}+\frac{y^2}{N^2}}(N(\alpha+1),l_M+1)=1-\frac{(2y(\alpha+1))^{l_M+1}}{\Gamma(l_M+1)}\int_0^1 s^{l_m}\,\mathrm{e}^{-2y(\alpha+1)s}\,ds.$$

Finally using $\frac{1}{(1-(1-\frac{2y}{N}))^2}\sim\frac{N^2}{(2y)^2}$ gives the limiting expression for the mean eigenvalue density. For the correlation kernel the bulk scaling limit is given by $z_k=(1-\frac{y_k}{N})\,\mathrm{e}^{i(\phi_0+\frac{\phi_k}{N})}$. First note that:

$$\begin{aligned}z_k\bar{z}_l&=(1-\frac{y_k}{N})\,\mathrm{e}^{i(\phi_0+\frac{\phi_k}{N})}(1-\frac{y_l}{N})\,\mathrm{e}^{i(\phi_0+\frac{\phi_l}{N})}\\ &\sim(1-\frac{y_k+y_l}{N})\,\mathrm{e}^{\frac{i}{N}(\phi-\phi_l)}\sim 1-\frac{y_k+y_l}{N}-i\frac{\phi_k-\phi_l}{N}.\end{aligned} \tag{3.2.124}$$

Furthermore:

$$\begin{aligned}&\frac{\big(1-(1-\frac{y_k}{N})^2\big)^{\frac{l_M-1}{2}}\big(1-(1-\frac{y_l}{N})^2\big)^{\frac{l_M-1}{2}}}{\big[1-(1-\frac{y_k}{N})(1-\frac{y_l}{N})\big]^{l_M+1}}\sim\frac{N^{l_M+1}\big(\frac{2y_k}{N}-\frac{y_k^2}{N^2}\big)^{\frac{l_M-1}{2}}\big(\frac{2y_l}{N}-\frac{y_l^2}{N^2}\big)^{\frac{l_M-1}{2}}}{\big(y_k+y_l+i(\phi_k-\phi_l)\big)^{l_M+1}}\\ &\sim N^2\frac{\big(2\sqrt{y_ky_l}\big)^{l_M-1}}{\big(y_k+y_l+i(\phi_k-\phi_l)\big)^{l_M+1}}.\end{aligned} \tag{3.2.125}$$

In addition for $I_1 = I_{\lambda_k\bar{\lambda}_l}(L, l_M+1)$ exploiting the beta function asymptotics yields:

$$1 - I_1 \sim \frac{\big((y_k+y_l+i(\phi_k-\phi_l))\alpha\big)^{l_M+1}}{\Gamma(l_M+1)} \int_0^1 s^{l_M}\, \mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))\alpha s}\, ds. \tag{3.2.126}$$

Similarly:

$$1 - I_2 \sim \frac{\big((y_k+y_l+i(\phi_k-\phi_l))(\alpha+1)\big)^{l_M+1}}{\Gamma(l_M+1)} \int_0^1 s^{l_M}\, \mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))(\alpha+1)s}\, ds. \tag{3.2.127}$$

Consequently the correlation kernel in the bulk in the limit of large matrix dimension is given by the following expression:

$$\begin{aligned} &K_N^{\mathrm{IndJacobi}}(z_k, z_l) \sim \frac{N^2}{\pi}\frac{\big(2\sqrt{y_k y_l}\big)^{l_M-1}}{\Gamma(l_M)}\times \\ &\left[\int_0^{\alpha+1} s^{l_M}\, \mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\, ds - \int_0^{\alpha} s^{l_M}\, \mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\, ds\right]. \end{aligned} \tag{3.2.128}$$

□

Almost square matrices and strong non-unitarity

In the asymptotic regime of almost square matrices with strong non-unitarity the rectangularity parameter is kept fixed $L = O(1)$, while $K = kN$ grows proportionally with matrix size. This implies the relations $l_M = (k-1)N - L$ as well as $l_N = (k-1)N$. Set $\mu_2 = \frac{1}{k}$.

In the limit of large matrix dimensions the eigenvalues are distributed across a disk around the origin with radius $r^{\mathrm{out}} = \sqrt{\mu_2}$. Furthermore in the limit of large matrix dimensions the n-point correlation functions after unfolding likewise exhibit universal behavior in the bulk on the support of the eigenvalue density, while at the origin the complex induced Ginibre correlation kernel at the origin is recaptured. Thus the limiting correlation kernel of the induced Ginibre ensemble in the regime of almost square matrices is found.

Theorem 3.2.17. *In the regime of almost square matrices with strong non-unitarity, $L = O(1)$ fixed and $n - N = N\beta$ in the limit of large matrix dimension the mean eigenvalue density is given by:*

$$\lim_{N\to\infty} \frac{1}{l_M} R_1^{IndJacobi}(z) = \frac{1}{\pi}\frac{1}{(1-|z|^2)^2}\Theta\big(|z| - \sqrt{\mu_2}\,\big). \tag{3.2.129}$$

At the edge $z^{out} = \left(\sqrt{\mu_2} + \frac{\xi}{\sqrt{l_M}}\right) \mathrm{e}^{i\phi}$ *of the eigenvalue support:*

$$\lim_{N\to\infty} \rho_N^{IndJacobi}\big(z^{out}\big) = \pi\rho^{IndJacobi}\big(\sqrt{\mu_2}\big)\frac{1}{2\pi}\operatorname{erfc}\left(\sqrt{2\pi\rho^{IndJacobi}\big(\sqrt{\mu_2}\big)}\xi\right)$$
$$= \frac{1}{2\pi}\frac{1}{(1-\mu_2)^2}\operatorname{erfc}\left(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\right). \tag{3.2.130}$$

Furthermore for the correlation kernel at the bulk set $z_k = u + \frac{s_k}{\sqrt{n+L}}$, $k = 1, \ldots, N$, *where* $u, s_k \in \mathbb{C}$ $0 < |u| < \sqrt{\mu_2} < \infty$. *Then:*

$$\lim_{N\to\infty} \frac{1}{l_M} K_N^{IndJacobi}(z_k, z_l) = \frac{1}{\pi}\frac{1}{(1-|u|^2)^2}\,\mathrm{e}^{-\frac{1}{1-|u|^2}\left(\frac{1}{2}|s_k|^2 + \frac{1}{2}|s_l|^2 - s_k\bar{s}_l\right)}. \tag{3.2.131}$$

At the origin $u = 0$ *with* $z_k = \frac{s_k}{\sqrt{n+L}}$ *the correlation kernel is given by:*

$$K_{origin}^{IndJacobi}(s_k, s_l) = \lim_{N\to\infty}\frac{1}{l_M}K_N^{IndJacobi}(z_k, z_l)$$
$$= \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|s_k|^2 + \frac{1}{2}|s_l|^2 - s_k\bar{s}_l}\,\frac{\gamma(s_k\bar{s}_l, L)}{\Gamma(L)}. \tag{3.2.132}$$

Proof. Using theorem A.2.3 yields:

$$\lim_{N\to\infty} I_{|z|^2}(L, (k-1)N - L + 1) = 1 \tag{3.2.133}$$

as well as:

$$\lim_{N\to\infty} I_{|z|^2}(N+L, (k-1)N - L + 1) = \Theta\big(\sqrt{\mu_2} - |z|\big). \tag{3.2.134}$$

Thus the mean eigenvalue density is supported on a disk around the origin with radius $\sqrt{\mu_2}$. Applying theorem A.2.4 gives the asymptotic behavior of the mean eigenvalue density close to the outer edge. While the proof of theorem 3.2.15 together with A.2.6 gives the asymptotic correlation kernel in the bulk. At the origin with scaling $z_k = \frac{s_k}{\sqrt{n+L}}$ the kernel can be written as:

$$K_N^{\text{IndJacobi}}(z_k, z_l) = \frac{l_M}{\pi}\frac{\left[(1-\frac{|s_k|^2}{l_M})(1-\frac{|s_l|^2}{l_M})\right]^{\frac{l_M-1}{2}}}{\left(1-\frac{s_k\bar{s}_l}{l_M}\right)^{L_M+1}} \times$$
$$\left[I_{\frac{s_k\bar{s}_l}{l_M}}(L, l_M) - I_{\frac{s_k\bar{s}_l}{l_M}}(M, l_M)\right]. \tag{3.2.135}$$

It follows that:

$$\frac{\left[(1-\frac{|s_k|^2}{l_M})(1-\frac{|s_l|^2}{l_M})\right]^{\frac{l_M-1}{2}}}{\left(1-\frac{s_k\bar{s}_l}{l_M}\right)^{L_M+1}} \sim \mathrm{e}^{-\frac{1}{2}|s_k|^2 - \frac{1}{2}|s_l|^2 + s_k\bar{s}_l}. \tag{3.2.136}$$

Furthermore:

$$\begin{aligned} I_{\frac{s_k \bar{s}_l}{l_M}}(L, l_M) =& \frac{1}{B(L, l_M)} \int_0^{\frac{s_k \bar{s}_l}{l_M}} t^{L-1}(1-t)^{l_M} dt \\ \sim& \frac{l_M^L}{\Gamma(L)} \frac{1}{l_M^L} \int_0^{s_k \bar{s}_l} t^{L-1}\big(1 - \frac{t}{l_M}\big)^{l_M} dt \\ \sim& \frac{1}{\Gamma(L)} \int_0^{s_k \bar{s}_l} t^{L-1} \, \mathrm{e}^{-t} \, dt. \end{aligned} \tag{3.2.137}$$

Similarly:

$$\begin{aligned} I_{\frac{s_k \bar{s}_l}{l_M}}(M, l_M) =& \frac{1}{B(M, l_M)} \int_0^{\frac{s_k \bar{s}_l}{l_M}} t^{M-1}(1-t)^{l_M} dt \\ \sim& \frac{1}{B(M, l_M)} \frac{1}{l_M^L} \int_0^{s_k \bar{s}_l} t^{M-1} \, \mathrm{e}^{-t} \, dt. \end{aligned} \tag{3.2.138}$$

□

Almost square matrices and weak non-unitarity

Finally in the regime of almost square matrices with weak non-unitarity the rectangularity parameter $L = O(1)$ as well as $l_M = O(1)$ are both kept fixed. As a result the eigenvalues are again distributed in the vicinity of the unit circle. In the limit of large matrix dimensions the mean eigenvalue density of truncations of random unitary matrices in the regime of weak non-unitarity is recovered. Equally in the bulk of the eigenvalue support the eigenvalue correlations coincide with the correlations found in [SŻ00, KSŻ10] for truncations of unitary matrices in the regime of weak non-unitarity. At the origin the limiting correlation kernel of the complex induced Ginibre ensemble is found (see equation (3.1.71)).

Theorem 3.2.18. *Set $z = (1 - \frac{y}{N})\,\mathrm{e}^{i\phi}$. Then in the regime of almost square matrices with weak non-unitarity in the limit of large matrix dimensions the mean eigenvalue density is given by:*

$$\lim_{N \to \infty} \frac{1}{N^2} R_1^{IndJacobi}\big((1 - \frac{y}{N})\,\mathrm{e}^{i\phi}\big) = \frac{1}{\pi} \frac{\Gamma(2y, l_M + 1)}{\Gamma(l_M)}. \tag{3.2.139}$$

For the correlation kernel at the bulk we scale $z_k = (1 - \frac{y_k}{N})\,\mathrm{e}^{i(\phi_0 + \frac{\phi_k}{N})}$, then:

$$\begin{aligned} &\lim_{N \to \infty} \frac{1}{N^2} K_N^{IndJacobi}(z_k, z_l) \\ =& \frac{1}{\pi} \frac{\big(2\sqrt{y_k y_l}\big)^{l_M - 1}}{\Gamma(l_M)} \int_0^1 s^{l_M} \, \mathrm{e}^{-(y_k + y_l + i(\phi_k - \phi_l))s} \, ds. \end{aligned} \tag{3.2.140}$$

Proof. From the eigenvalue scaling we obtain:

$$\lim_{N\to\infty} I_{|z|^2}(L, l_M+1) = 1. \tag{3.2.141}$$

and theorem A.2.2 yields:

$$\lim_{N\to\infty} I_{1-\frac{2y}{N}+\frac{y^2}{N^2}}(N(\alpha+1), l_M+1) = 1 - \frac{(2y)^{l_M+1}}{\Gamma(l_M+1)}\int_0^1 s^{l_m} e^{-2ys} ds. \tag{3.2.142}$$

As a result we recover the same asymptotic behavior for the mean eigenvalue density as in the case of square truncations. In the correlation kernel we again scale $z_k = (1-\frac{y_k}{N})\,\mathrm{e}^{i(\phi_0+\frac{\phi_k}{N})}$ for $k = 1, \ldots, N$ and note as before:

$$z_k\bar{z}_l \sim 1 - \frac{y_k+y_l}{N} - i\frac{\phi_k-\phi_l}{N}. \tag{3.2.143}$$

Then:

$$\lim_{N\to\infty} I_{z_k\bar{z}_l}(L, l_M+1) = 1, \tag{3.2.144}$$

as well as:

$$\begin{aligned}&\lim_{N\to\infty} I_{1-\frac{2y}{N}+\frac{y^2}{N^2}}(N(\alpha+1), l_M+1)\\ =&1 - \frac{\big(y_k+y_l+i(\phi_k-\phi_l)\big)^{l_M+1}}{\Gamma(l_M+1)}\int_0^1 s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds,\end{aligned} \tag{3.2.145}$$

which proves our result. □

3.3 Summary of results

3.3.1 The complex induced spherical ensemble

- The eigenvalue jpdf of a complex induced spherical matrix:

$$p_{\text{IndSpherical},2}(\lambda_1, \ldots, \lambda_N) \propto \prod_{j<k}|\lambda_k-\lambda_j|^2 \prod_{j=1}^N \frac{|\lambda_j|^{2L}}{(1+|\lambda_j|^2)^{n+L+1}}. \tag{3.3.1}$$

- The finite N mean eigenvalue density of a complex induced spherical matrix:

$$R_1^{\text{IndSpherical}}(z) = \frac{1}{\pi}\frac{n+L}{\left(1+|z|^2\right)^2}\left[I_{\frac{|z|^2}{1+|z|^2}}(L,n) - I_{\frac{|z|^2}{1+|z|^2}}(M, n-N)\right]. \tag{3.3.2}$$

- Eigenvalue support in the four distinct asymptotic regimes:

Strong rectangularity, strong spherical component: Annulus
Strong rectangularity, weak spherical component: complex plane without disk around origin
Almost square, strong spherical component: disk about origin
Almost square, weak spherical component: whole complex plane

- Limiting correlation kernel in the bulk in the two regimes of strong rectangularity: complex Ginibre after unfolding, see theorem 3.1.13.

- Limiting correlation kernel in the bulk in the two regimes of almost square matrices: complex Ginibre after unfolding, see theorem 3.1.13.
  Limiting correlation kernel at the origin in the two regimes of almost square matrices:

$$K^{\mathrm{IndGin}}_{\mathrm{origin}}(\lambda_k, \lambda_l) = \frac{1}{\pi}\, \mathrm{e}^{-\frac{1}{2}|\lambda_k|^2 - \frac{1}{2}|\lambda_l|^2 + \lambda_k \bar{\lambda}_l}\, \frac{\gamma(L, \lambda_k \bar{\lambda}_l)}{\Gamma(L)}. \tag{3.3.3}$$

3.3.2 The complex induced Jacobi ensemble

- The eigenvalue jpdf of a complex induced Jacobi matrix:

$$p_{\mathrm{IndJacobi},2}(\lambda_1, \ldots, \lambda_N) \propto \prod_{j<k} |\lambda_k - \lambda_j|^2 \prod_{j=1}^{N} |\lambda_j|^{2L} (1 - |\lambda_j|^2)^{l_M - 1}. \tag{3.3.4}$$

- The finite N mean eigenvalue density of a complex induced Jacobi matrix:

$$R_1^{\mathrm{IndJacobi}}(z) = \frac{1}{\pi} \frac{l_M}{\left(1 - |z|^2\right)^2} \cdot \left[I_{|z|^2}(L, l_M + 1) - I_{|z|^2}(M, l_M + 1) \right]. \tag{3.3.5}$$

- Eigenvalue support in the four distinct asymptotic regimes:

  Strong rectangularity, strong non-unitarity: Annulus
  Strong rectangularity, partially weak non-unitarity: eigenvalues distributed close to the unit circle
  Almost square, strong non-unitarity: disk around origin with radius less than one
  Almost square, weak non-unitarity: eigenvalues distributed close to the unit circle

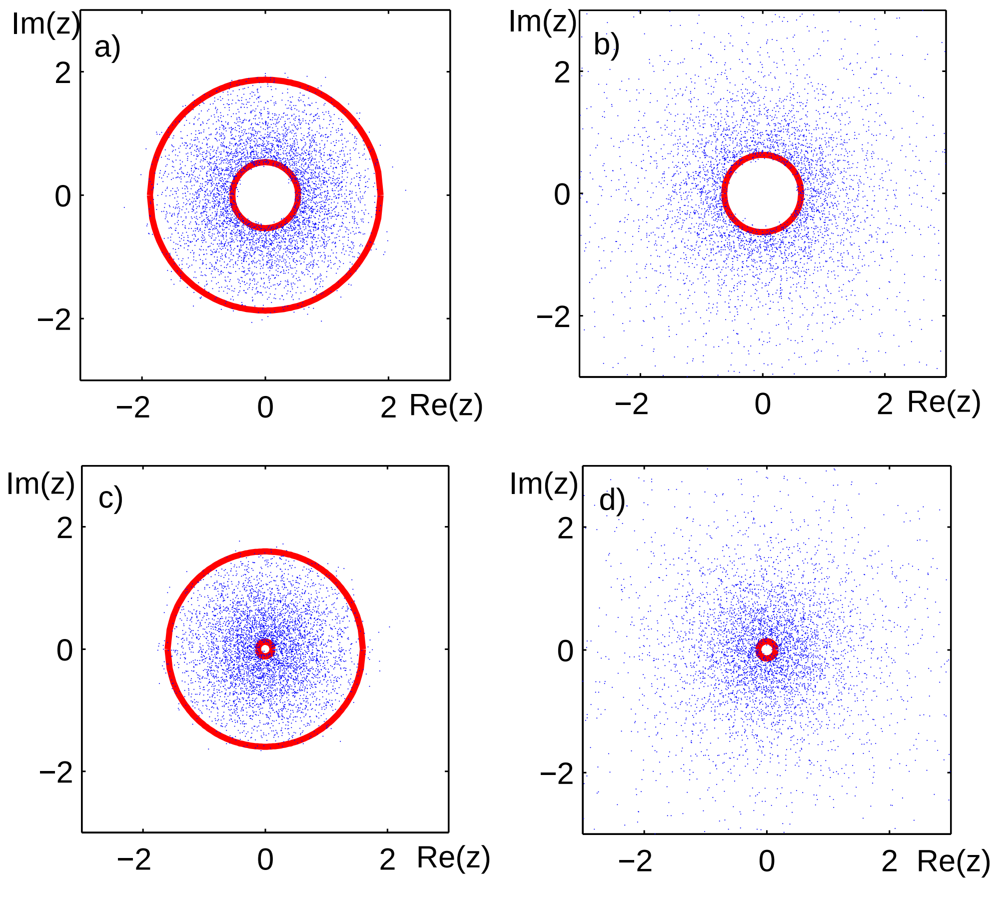


Figure 3.2: Spectra of matrices pertaining to the induced spherical ensemble of complex matrices for dimension $N = 100$ and a) $L = 40$, $n - N = 40$, b) $L = 40$, $n - N = 0$, c) $L = 2$, $n - N = 40$, d) $L = 2$, $n - N = 2$. Each plot consists of data from 50 independent realizations. The circles of radius $r_{\mathrm{in}} = \sqrt{L/n}$ (inner one) and $r_{\mathrm{out}} = \sqrt{M/(N-n)}$ (outer one) are depicted to guide the eye.

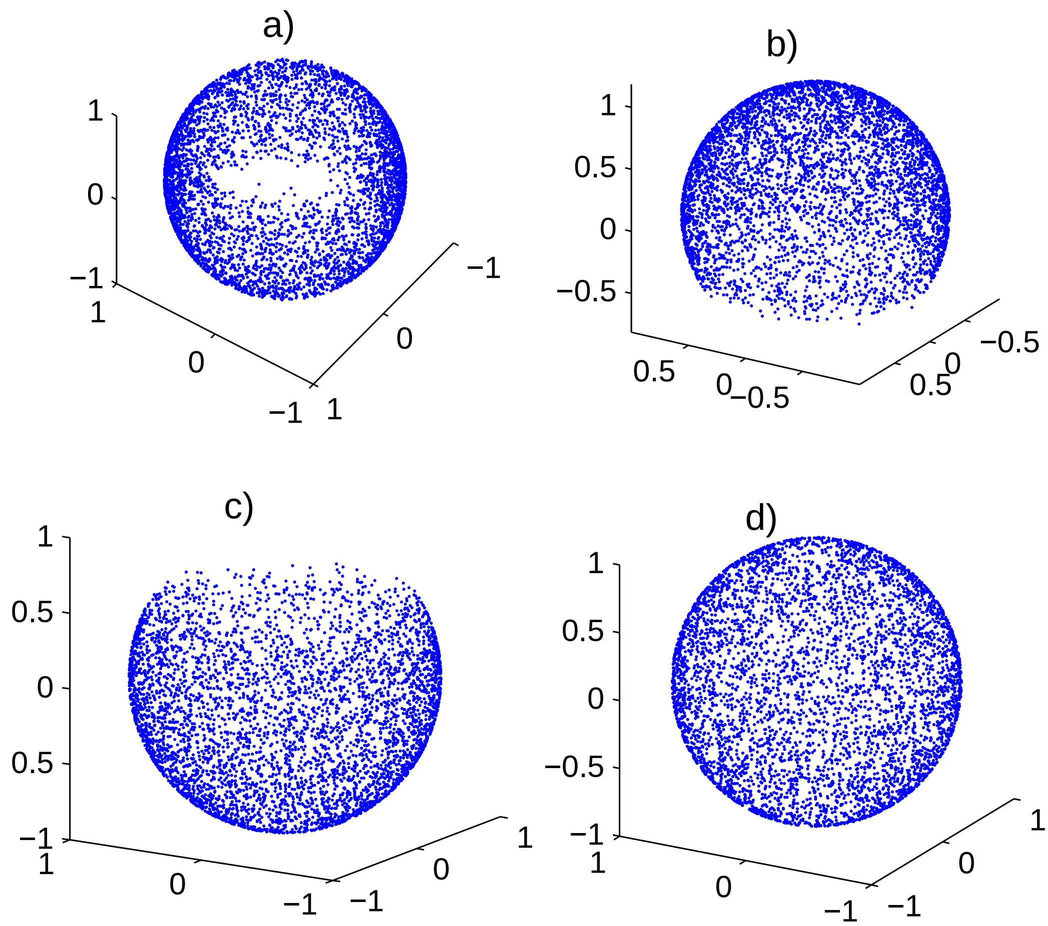


Figure 3.3: Spectra of matrices pertaining to the induced spherical ensemble of complex matrices for dimension $N = 100$ and a) $L = 40$, $n - N = 40$, b) $L = 40$, $n-N = 0$, c) $L = 2$, $n-N = 40$, d) $L = 2$, $n-N = 2$ after inverse stereographical projection to the sphere.

- Limiting correlation kernel in the bulk in the regime of strong rectangularity and strong non-unitarity: complex Ginibre after unfolding, see theorem 3.1.13.

- Limiting correlation kernel in the bulk in the regime of strong rectangularity and partially weak non-unitarity, see theorem 3.2.16:

$$\lim_{N\to\infty}\frac{1}{N^2}K_N^{\text{IndJacobi}}(z_k,z_l) = \frac{1}{\pi}\frac{\left(2\sqrt{y_k y_l}\right)^{l_M-1}}{\Gamma(l_M)}\Big[\int_0^{\alpha+1} s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds - \int_0^{\alpha} s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds\Big]. \tag{3.3.6}$$

- Limiting correlation kernel in the bulk in the regime of almost square matrices and strong non-unitarity: complex Ginibre after unfolding, see theorem 3.1.13.
  Limiting correlation kernel at the origin in the regime of almost square matrices and strong non-unitarity:

$$K_{\text{origin}}^{\text{IndGin}}(\lambda_k,\lambda_l) = \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|\lambda_k|^2-\frac{1}{2}|\lambda_l|^2+\lambda_k\bar{\lambda}_l}\,\frac{\gamma(L,\lambda_k\bar{\lambda}_l)}{\Gamma(L)}. \tag{3.3.7}$$

- Limiting correlation kernel in the bulk in the regime of almost square matrices and weak non-unitarity: truncations of random unitary matrices, see theorem 3.2.18:

$$\lim_{N\to\infty}\frac{1}{N^2}K_N^{\text{IndJacobi}}(z_k,z_l) = \frac{1}{\pi}\frac{\left(2\sqrt{y_k y_l}\right)^{l_M-1}}{\Gamma(l_M)}\int_0^1 s^{l_M}\,\mathrm{e}^{-(y_k+y_l+i(\phi_k-\phi_l))s}\,ds. \tag{3.3.8}$$

3.4 Application: The two-dimensional one-component plasma

3.4.1 The one-dimensional two-component plasma and random matrix theory

The two-dimensional one-component plasma is an equilibrium statistical mechanical system consisting of N mobile particles each of charge $+1$ and a smeared out neutralizing background. The particles are confined to a two-dimensional surface and the charge densities interact through the solution of the two-dimensional Poisson equation on the surface.

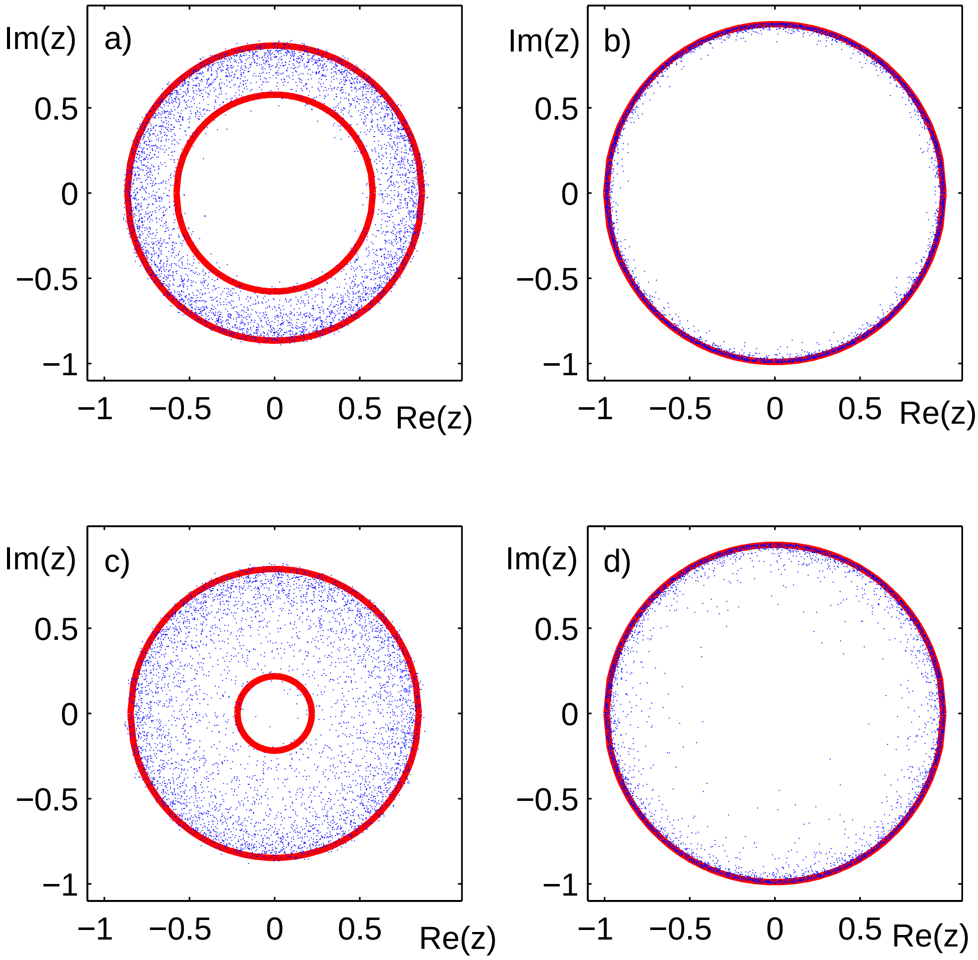


Figure 3.4: Spectra of matrices pertaining to the induced Jacobi ensemble of complex matrices for dimension $N = 100$ and a) $L = 20$, $l_M = 40$, b) $L = 20$, $l_M = 2$, c) $L = 2$, $l_M = 20$, d) $L = 2$, $l_M = 2$. Each plot consists of data from 50 independent realizations. The circles of radius $r_{\rm in} = \sqrt{L/K}$ (inner one) and $r_{\rm out} = \sqrt{M/K}$ (outer one) are depicted to guide the eye.

In certain cases it can be shown that the Boltzmann factor for the plasma at inverse temperature $\beta = 2$ coincides with the eigenvalue joint probability function of a given non-hermitian random matrix ensemble. The best known example of such an analogy is between the one-component plasma on a disk of radius $\sqrt{N}$ and the complex Ginibre ensemble, from definition 2.0.19 [AJ81, Gin65]. If the eigenvalues of a Ginibre matrix are then confined to the disk of radius $\sqrt{N}$ then the eigenvalue jpdf of the complex Ginibre ensemble coincides with the Boltzmann factor of the one-component plasma. Since then two other examples of this analogy were presented in [FN08]. It was shown that the eigenvalue jpdf of a complex spherical matrix from definition 2.0.24 with $n = N$ coincides with the Boltzmann factor of the two-component plasma on the sphere at inverse temperature $\beta = 2$ after a stereographical projection of the latter. An additional example is provided by the analogy of the eigenvalue jpdf of a truncated unitary matrix and the one-component plasma at $\beta = 2$ on the pseudo-sphere after a stereographical projection of the latter onto the Poincaré disk.

3.4.2 The two-dimensional one-component plasma on a spherical annulus [FF11]

Consider a sphere S of radius R, and let $0 \leq \theta \leq \pi$ refer to the usual azimuthal angle, and $0 \leq \phi \leq 2\pi$ refer to the polar angle. For two points (θ, ϕ) and (θ', ϕ') on the sphere, let α refer to their relative angle when considered as vectors in $\mathbb{R}^3$. Furthermore mark two circles on the sphere corresponding to the azimuthal angles θ_Q and $\pi - \theta_q$, with $0 < \theta_Q < \pi - \theta_q < \pi$. We call the surface that is defined between the two circles on the disk a spherical annulus. In addition we denote the area of the spherical cap including the north pole above the angle θ_Q with $A_{[0,\theta_Q]}$. Similarly we denote the area below the angle $\pi - \theta_q$ and thus including the south pole with $A_{[\pi-\theta_q,\pi]}$.

The plasma system is now specified as follows. Inside the spherical annulus there are N mobile charges of charge $+1$ and a uniform neutralizing background. In addition both spherical caps $A_{[0,\theta_Q]}$,$A_{[\pi-\theta_q,\pi]}$ are endowed with a positive uniform density. It is useful to parameterize the spherical annulus by introducing Q and q such that

$$\frac{A_{[0,\theta_Q]}}{4\pi R^2} = \frac{Q}{1+q+Q}, \qquad \frac{A_{[\pi-\theta_q,\pi]}}{4\pi R^2} = \frac{q}{1+q+Q}. \tag{3.4.1}$$

Then the uniform charge density throughout the sphere is given by:

$$-\rho_b := -\frac{N}{4\pi R^2}(1+Q+q). \tag{3.4.2}$$

In addition the uniform charge

$$-\bar{\rho}_b := -\frac{N}{4\pi R^2}(Q+q). \tag{3.4.3}$$

is distributed across the spherical caps. The uniform and discrete charges interact via the solution of the two-dimensional Poisson equation.

$$\nabla^2_{\theta,\phi}\Phi = -2\pi\delta_S((\theta,\phi),(\theta',\phi')) + \frac{1}{2R^2} \tag{3.4.4}$$

where $\delta_S((\theta,\phi),(\theta',\phi'))$ is the delta function on the sphere. The solution of the Poisson equation can be written in terms of the Cayley-Klein parameters

$$u := \cos(\theta/2)e^{i\phi/2}, \qquad v := -i\sin(\theta/2)e^{-i\phi/2} \tag{3.4.5}$$

as follows [FF11]

$$\Phi((\theta,\phi),(\theta',\phi')) = -\log(2R|u'v - uv'|). \tag{3.4.6}$$

The total potential energy U of the plasma system can be determined by computing the particle-particle, particle-background and background-background interaction. All in all from [FF11] the Boltzmann factor $e^{-\beta U}$ for the plasma system is equal to

$$\Big(\frac{1}{2R}\Big)^{N\beta/2} e^{-\beta K_N} \prod_{l=1}^{N} |v_l|^{\beta Q N}|u_l|^{\beta q N} \prod_{1\le j<k\le N} |u_k v_j - u_j v_k|^{\beta}, \tag{3.4.7}$$

where

$$\begin{aligned} K_N :=& \frac{N^2}{4}\Big(-(1+Q+q) + 2(1+Q+q)\log\frac{1}{1+Q+q} + (1+q)^2\log(1+q) \\ &+ (1+Q)^2\log(1+Q) - Q^2\log q - q^2\log Q\Big). \end{aligned} \tag{3.4.8}$$

An inverse stereographical projection transformation as well as a rescaling yields the following for the Boltzmann factor of the one-component plasma

$$\prod_{l=1}^{N}\Big(\frac{|\tilde{z}_l|^2}{1+|\tilde{z}_l|^2}\Big)^{\beta Q N/2} \frac{1}{(1+|\tilde{z}_l|^2)^{\beta q N/2+2+\beta(N-1)/2}} \prod_{1\le j<k\le N} |\tilde{z}_j - \tilde{z}_k|^{\beta} \tag{3.4.9}$$

Thus an analogy between the Boltzmann factor of the one-component plasma on a spherical annulus and the eigenvalue jpdf of a complex induced spherical random matrix was found.

3.4.3 Future work

Naturally the question arises, whether it would be possible to find analogies between the complex induced Ginibre ensemble, as well as the complex induced Jacobi ensemble and some plasma system. In the case of the complex induced Ginibre ensemble in the regime of strong rectangularity in the limit of large matrix dimensions the eigenvalues are to leading order uniformly distributed on an annulus. It should be possible to construct an one-component plasma system confined to the annulus with charges being repelled from the origin, whose Boltzmann factor coincides with the eigenvalue jpdf of the induced Ginibre ensemble for $\beta = 2$. In the case of the complex induced Jacobi ensemble it might be possible to construct an one-component plasma system consisting of charges confined to an pseudo-spherical annulus. It can easily be believed that after some projection the eigenvalues of the complex induced Jacobi ensemble are uniformly distributed on some part of the pseudo-sphere. Similarly it is most likely possible to find an analogy between the eigenvalue jpdf of the complex induced Jacobi ensemble and the Boltzmann factor of this plasma system for $\beta = 2$. Further work would be needed to verify these conjectures.

Chapter 4

Real induced non-hermitian random matrix ensembles

4.1 The real induced Ginibre ensemble

Again the simplest example of a real induced random matrix is provided by applying the real inducing procedure to a rectangular real Ginibre matrix. From theorem 2.2.1:

Definition 4.1.1. *The real induced Ginibre ensemble on the space of $N \times N$ matrices is specified by the matrix measure:* $\mathrm{d}\,\mu^{Induced}_{Ginibre,1} = P^{Induced}_{Ginibre,1}(G)(dG)$ *with*

$$P^{Induced}_{Ginibre,1}(G) = c^{Induced}_{Ginibre,1} \det(GG^T)^L \, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(GG^T)}, \qquad L = M - N \geq 0 \tag{4.1.1}$$

and

$$c^{Induced}_{Ginibre,1} = \pi^{-\frac{1}{2}N^2} 2^{-\frac{1}{2}N^2 + \frac{1}{2}NL} \prod_{j=1}^{N} \frac{\Gamma(\frac{j}{2})}{\Gamma\left(\frac{j+L}{2}\right)} \,. \tag{4.1.2}$$

Clearly setting the parameter $L = 0$ leads back to the real Ginibre ensemble [Gin65]. The rectangularity parameter L denotes the mismatch in dimensions of the rectangular Ginibre matrix, used to generate the induced Ginibre ensemble. The subsequent analysis extends verbatim to non-negative real values of L. As in the case of the complex induced Ginibre ensemble no matrix interpretation is known for non-integer values of L. Note that for real square matrices G and Haar distributed U:

$$G \sim U\sqrt{G^T G}. \tag{4.1.3}$$

4.1.1 The joint distribution of eigenvalues

The difficulty in deriving the joint probability density function for real asymmetric matrices is due to the fact that there is a non-zero probability $p_{N,k}^{\mathrm{IndGin}}$ for the matrix G to have k real eigenvalues. In the following it is assumed that G has k real ordered eigenvalues: $\lambda_1 \geq \ldots \geq \lambda_k$, while $l = \frac{N-k}{2}$ denotes the number of complex conjugate eigenvalue pairs $z_1, \bar{z}_1, \ldots, z_l, \bar{z}_l$ ordered by their real part. In the case of two complex eigenvalues with identical real part the eigenvalue pairs are ordered by their imaginary part.

As a consequence the eigenvalue jpdf decomposes into a sum of probability densities $P_{N,k,l}^{\mathrm{IndGin}}(\lambda_1, \ldots, \lambda_k, z_1, \ldots, z_l)$, corresponding to having k real eigenvalues and l pairs of complex conjugate eigenvalues. In order for $P_{N,k,l}^{\mathrm{IndGin}}$ to be non-zero k must be even, as it is assumed that N is even.

The derivation of the eigenvalue jpdf follows [Ede97], see [LS91, SW08, KS09] for alternative derivations. In order to change variables from the entries of G to the eigenvalues of G and some auxiliary variables the real Schur decomposition from (1.3.15) is employed: $G = QRQ^T$, where $Q \in \mathbb{R}^{N \times N}$ is an orthogonal matrix, whose first row is chosen to be non-negative and the matrix $R \in \mathbb{R}^{N \times N}$ is block triangular of the form:

$$R = \begin{pmatrix} \lambda_1 & \cdots & r_{1k} & r_{1,k+1} & \cdots & r_{1,N} \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \lambda_k & r_{k,k+1} & \cdots & r_{k,N} \\ 0 & \cdots & 0 & Z_1 & \cdots & r_{k+1,N} \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & Z_l \end{pmatrix} = \begin{pmatrix} \Lambda & S \\ 0 & Z \end{pmatrix}. \tag{4.1.4}$$

Here Λ is triangular containing the real eigenvalues $\lambda_1, \ldots, \lambda_k$ of G on its diagonal and Z is block triangular containing the 2×2 blocks:

$$Z_j = \begin{pmatrix} x_j & b_j \\ -c_j & x_j \end{pmatrix}, \quad b_j c_j > 0, \quad b_j \leq c_j \quad \text{and} \quad y_j = \sqrt{b_j c_j} \tag{4.1.5}$$

on its block diagonal. The complex conjugate eigenvalue pairs are given by: $z_m = x_m + iy_m$ and $\bar{z}_m = x_m - iy_m$ for $m = 1, \ldots, l$. The Jacobian of this change of variable is computed in theorem 1.3.17:

$$|J| = 2^l \big|\Delta\big(\{\lambda_j\}_{j=1,\dots,k} \cup \{z_m, \bar{z}_m\}_{m=1,\dots,l}\big)\big| \prod_{i>k} (b_i - c_i), \tag{4.1.6}$$

with $\Delta\big(\{z_p\}_{p=1,\dots,n}\big) := \prod_{i<j}(z_j - z_i)$ denoting the Vandermonde determinant. Consequently we arrive at the relation:

$$P^{\text{Induced}}_{\text{Ginibre},1}(G)(dG) = c^{\text{Induced}}_{\text{Ginibre},1}|J| \prod_{j=1}^{k} |\lambda_j|^L \prod_{m=1}^{l} (x_m^2 + b_m c_m)^L \times$$
$$\mathrm{e}^{-\frac{1}{2}\sum_{j=1}^k \lambda_j^2 - \frac{1}{2}\sum_{i<j} r_{ij}^2 - \sum_{j=1}^l (x_j^2 + \frac{b_j^2}{2} + \frac{c_j^2}{2})}(d\Lambda)(dR)(Q^T dQ), \tag{4.1.7}$$

where $(Q^T dQ)$ is taken as in definition 1.3.8. In addition another change of variable is necessary from the entries x_j, b_j, c_j of the matrix blocks Z_j to the real and imaginary part x_j, y_j of the complex conjugate eigenvalue pairs and an auxiliary variable δ_j. The change of variable is performed in the following way:

$$\text{Set} \quad b_j = \frac{1}{2}\left(\delta_j + \sqrt{\delta_j^2 + 4y_j^2}\right) \quad \text{and} \quad c_j = \frac{1}{2}\left(-\delta_j + \sqrt{\delta_j^2 + 4y_j^2}\right), \tag{4.1.8}$$

which implies $y_j = \sqrt{b_j c_j}$ and $\delta_j = b_j - c_j$. The Jacobian of this second change of variables can easily be determined:

$$|\bar{J}| = \frac{4y_j}{\sqrt{\delta_j^2 + 4y_j^2}}. \tag{4.1.9}$$

Integrating out the auxiliary variables δ_j for $j = 1, \dots, m$ and r_{ij} as well as using $\mathrm{Vol}(O[N]) = \frac{\pi^{\frac{1}{4}N(N+1)}}{\prod_{j=1}^N \Gamma(\frac{j}{2})}$ finally yields the partial eigenvalue joint probability density function:

Theorem 4.1.2. *The eigenvalue jpdf of a matrix $G \in \mathbb{C}^{N\times N}$ pertaining to the real induced Ginibre ensemble, with parameter L and k real eigenvalues as well as l complex conjugate eigenvalue pairs, is given by:*

$$P^{IndGin}_{N,k,l}(\lambda_1, \dots, \lambda_k, z_1, \dots, z_l) = c^{IndGin}_{N,k,l} \big|\Delta\big(\{\lambda_j\}_{j=1}^k \cup \{z_m, \bar{z}_m\}_{m=1}^l\big)\big| \times$$
$$\prod_{j=1}^k w_{IndGin,1}(\lambda_j) \prod_{m=1}^l \mathrm{Im}(z_m) w_{IndGin,1}(z_m) w_{IndGin,1}(\bar{z}_m), \tag{4.1.10}$$

where

$$w_{IndGin,1}(z) = z^L \,\mathrm{e}^{-\frac{1}{2}z^2} \left(\mathrm{erfc}\left(\sqrt{2}\,\mathrm{Im}(z)\right)\right)^{\frac{1}{2}} \tag{4.1.11}$$

$$c_{N,k,l}^{IndGin} = \frac{2^{2l-\frac{1}{4}N(N+1)}\pi^{-NL}}{\prod_{j=1}^{N}\Gamma(\frac{L+j}{2})} \tag{4.1.12}$$

and $\lambda_j \in \mathbb{R}$ *for* $j = 1, \dots, k$ *and* $z_m \in \mathbb{C}_+$ *for* $m = 1, \dots, l$. *Integrating the partial eigenvalue jpdf* $P_{N,k,l}^{IndGin}$ *over* $\mathbb{R}^k \times \mathbb{C}_+^{2l}$ *gives* $p_{N,k}^{IndGin}$.

Thus the inducing procedure results in the additional factor $\prod_{j=1}^{N} |\lambda_j|^{2L}$ in the symmetrized eigenvalue jpdf of the real induced Ginibre ensemble. As a consequence the probability of finding eigenvalues close to zero is small, which result in a repulsion from the origin. The larger the mismatch of dimension in the original rectangular Ginibre matrix, used to generate the complex induced Ginibre ensemble, the stronger the repulsion of eigenvalues away from the origin.

4.1.2 The method of skew-orthogonal polynomials

The aim of the following three sections is to arrive at a closed form expression for the correlation functions of the real induced Ginibre ensemble. For this purpose the method of skew-orthogonal polynomials is introduced. The method of skew-orthogonal polynomials is the real equivalent of the method of orthogonal polynomials employed in chapter 3. Starting point for the application of the method of skew-orthogonal polynomials is the generalized partition function, which we define below.

Definition 4.1.3. *The partial generalized partition funtion of a real asymmetric random matrix ensemble is defined as follows:*

$$\begin{aligned} Z_{N,k,l}[u,v] = \int_{\mathbb{R}} d\lambda_1 \cdots \int_{\mathbb{R}} d\lambda_k \prod_{j=1}^{k} u(\lambda_j) \int_{\mathbb{C}_+} dz_1 \cdots \int_{\mathbb{C}_+} dz_l \prod_{m=1}^{l} v(z_m) \\ \times P_{N,k,l}(\lambda_1, \dots, \lambda_k, z_1, \dots, z_l), \end{aligned} \tag{4.1.13}$$

while the generalized partition function is defined as:

$$Z_N[u,v] = \sum_{k=0, k\ even}^{N} Z_{N,k,l}[u,v]. \tag{4.1.14}$$

An important breakthrough in the theory of non-hermitian random matrices was achieved by Sinclair in [Sin07], who succeeded in expressing the generalized partition function of the real Ginibre ensemble using a Pfaffian representation.

The methods used in [Sin07] do not depend on the particular weight function. As a result,

Theorem 4.1.4 ([Sin07])**.** *Let $I \in \{IndGin, IndSpherical, IndJacobi\}$ then the generalized partition function of the induced familiy of real asymmetric matrices is given by:*

$$Z_N^I[u,v] = c_N^I \operatorname{Pfaff} U_q^{w_{I,1}}, \tag{4.1.15}$$

where $\{q_j\}_{j=0,1,\ldots}$ is a family of monic polynomials, c_N^I is the normalization of the eigenvalue jpdf $P_{I,1}^{Induced}$ and $U_q^{w_{I,1}}$ is a $N \times N$ anti-symmetric matrix with entries

$$U_q^{w_{I,1}}[j,m] = (q_j, q_k)^I \quad \text{for} \quad j,m = 1,\ldots,N \tag{4.1.16}$$

Furthermore $(-,-)^I$ denotes the skew-symmetric inner product:

$$\begin{aligned}
(f,g)^I &:= (f,g)^I_{\mathbb{R}} + (f,g)^I_{\mathbb{C}} \\
(f,g)^I_{\mathbb{R}} &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sgn}(y-x) f(x) g(y) u(x) w_{I,1}(x) u(y) w_{I,1}(y) dx dy \\
(f,g)^I_{\mathbb{C}} &:= 2i \int_{\mathbb{C}_+} v(z) w_{I,1}(z) w_{I,1}(\bar{z}) \left[f(z) g(\bar{z}) - g(z) f(\bar{z}) \right] dz.
\end{aligned}$$

where $w_{I,1}$ is the weight function of the respective real asymmetric ensemble.

Remark 4.1.5. *Note that the representation of the generalized partition function is seemingly independent of the number of real and complex-conjugate eigenvalues. In truth the number of real eigenvalues enters theorem 4.1.15 through the skew-inner product $(-,-)^I$, which consists of a real and a complex component.*

Again we are interested in the correlations between the eigenvalues. The first starting point are the (K', L', k', l')-partial correlation functions which are just the symmetrized marginal probability density functions of K' real eigenvalues and L' complex eigenvalue pairs in the case that the number of real eigenvalues is k' while the number of complex eigenvalues is l' with different normalization.

Definition 4.1.6. *The (K', L', k', l')-partial correlation functions of a real asymmetric random matrix ensemble are defined as:*

$$R_{(K',L',k',l')}(\lambda_1,\ldots,\lambda_{K'}, z_1,\ldots,z_{L'}) = \frac{k'!l'!2^{l'-L'}}{(k'-K')!(l'-L')!} \times \tag{4.1.17}$$

$$\int_{\mathbb{R}^{k'-K'}} \int_{\mathbb{C}_+^{2(l'-L')}} P_{N,k',l'}(\lambda_1,\ldots,\lambda_{k'}, z_1,\ldots,z_{l'}) d\lambda_{k'-K'+1} \ldots d\lambda_{k'} d^2 z_{l'-L'+1} \ldots d^2 z_{l'} \,.$$

The (K', L')-correlation functions which are the symmetrized marginals of K' real eigenvalues and L' complex eigenvalue pairs with different normalization constant then decompose into a disjoint sum of probability density corresponding to having k' real and l' complex eigenvalues. They are defined as follows:

Definition 4.1.7. *The (K', L')-correlation functions of a real asymmetric random matrix ensemble are defined as:*

$$R_{K',L'}(\lambda_1, \ldots, \lambda_{K'}, z_1, \ldots, z_{L'}) = \sum_{\substack{(K',L') \\ K' \le k', L' \le l'}} R_{(K',L',k',l')}(\lambda_1, \ldots, \lambda_{K'}, z_1, \ldots, z_{L'}) \,. \tag{4.1.18}$$

An important observation is that the correlation functions can be obtained through the generalized partition function through functional differentiation:

$$\frac{\partial^{k'-K'+l'-L'}}{\partial u(\lambda_{K'}) \cdots \partial u(\lambda_{k'}) \partial v(z_{L'}) \cdots \partial v(z_{l'})} Z_N[u, v]\Big|_{u=v=1} . \tag{4.1.19}$$

Another milestone in non-hermitian random matrix theory was reached by Borodin and Sinclair in [BS09], see [FN07, Som07, SW08] as well, who succeeded in expressing the (K', L')-correlation functions of real asymmetric matrices in closed from by using a Pfaffian kernel representation. Again their proof of theorem 4.1.8 does not depend on the choice of weight function in the eigenvalue jpdf.

Theorem 4.1.8 ([BS09])**.** *Let $I \in \{IndGin, IndSpherical, IndJacobi\}$ then the (K', L')-correlation functions of the induced familiy of real asymmetric matrices are given by:*

$$R^I_{K',L'}(\lambda_1, \ldots, \lambda_{K'}, z_1, \ldots, z_{L'}) = \mathrm{Pfaff} \begin{bmatrix} K^I_N(x_j, x_{j'}) & K^I_N(x_j, z_{m'}) \\ K^I_N(z_m, x_{j'}) & K^I_N(z_m, z_{m'}) \end{bmatrix}, \tag{4.1.20}$$

with the 2×2 matrix kernel:

$$K^I_N(v, v') := \begin{bmatrix} DS^I_N(v, v') & S^I_N(v, v') \\ -S^I_N(v, v') & IS^I_N(v, v') + \varepsilon(v, v') \end{bmatrix}, \tag{4.1.21}$$

where

$$DS_N^I(v,v') = 2\sum_{n=0}^{N-1}\sum_{n'=0}^{N-1} \mu_{n,n'}^I \tilde{q}_n^I(v)\tilde{q}_{n'}^I(v') \tag{4.1.22}$$

$$S_N^I(v,v') = 2\sum_{n=0}^{N-1}\sum_{n'=0}^{N-1} \mu_{n,n'}^I \tilde{q}_n^I(v)\tau_{n'}^I(v') \tag{4.1.23}$$

$$IS_N^I(v,v') = 2\sum_{n=0}^{N-1}\sum_{n'=0}^{N-1} \mu_{n,n'}^I \tau_n^I(v)\tau_{n'}^I(v') \tag{4.1.24}$$

$$\varepsilon(v,v') = \begin{cases} \frac{1}{2}\operatorname{sgn}(v-v'), & \textit{if} \quad v,v' \in \mathbb{R} \\ 0, & \textit{else} \end{cases} \tag{4.1.25}$$

with

$$\tilde{q}^I(v) := w^I(v)q^I(v) \tag{4.1.26}$$

$$\tau_j^I(v) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(y-v)\tilde{q}_j^I(y)dy, & \textit{if} \quad v \in \mathbb{R} \\ i\tilde{q}_j^I(v)\operatorname{sgn}(\operatorname{Im}(v)), & \textit{if} \quad v \in \mathbb{C}\backslash\mathbb{R} \end{cases} . \tag{4.1.27}$$

Furthermore $\mu_{n,n'}^I$ is the (n,n')-th entry of the matrix $\left(U_q^{w_{I,1}}\right)^{-T}$ from theorem 4.1.15 with $u = v = 1$. The sub scripts j and j' in equation (4.1.20) run from 1 to K' whilst m and m' run from 1 to L', so that the matrix inside the Pfaffian has the block structure with the top left and right bottom blocks being of size $2K' \times 2K'$ and $2L' \times 2L'$, respectively.

The entries of the Pfaffian kernel depend on the family of polynomials q_j^I. Choosing the appropriate polynomials in the Pfaffian kernel entries from equations (4.1.22)-(4.1.24), results in a particularly simple form of the kernel entries. Consequently,

Definition 4.1.9. *A family $\{q_j\}_{j=1,\ldots,}$of skew-orthogonal polynomials is said to be skew-orthogonal with respect to the skew-symmetric inner product $(-,-)$, if it satisfies*

$$(q_{2j}, q_{2k}) = (q_{2j+1}, q_{2k+1}) = 0 \tag{4.1.28}$$

$$(q_{2j}, q_{2k+1}) = -(q_{2j+1}, q_{2k}) = r_j\delta_{jk} \quad \text{for} \quad j,k = 0,1,\ldots\,. \tag{4.1.29}$$

Choosing the polynomials q_j^I in theorem 4.1.8 skew-orthogonal with respect to the skew-inner product $(-,-)^I$ gives the a particularly simple form of the kernel entries [BS09]:

$$DS_N^I(v,v') = 2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^I}\left[\tilde{q}_{2j}^I(v)\tilde{q}_{2j+1}^I(v') - \tilde{q}_{2j+1}^I(v)\tilde{q}^I{}_{2j}(v')\right] \tag{4.1.30}$$

$$S_N^I(v,v') = 2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^I}\left[\tilde{q}_{2j}^I(v)\tau_{2j+1}^I(v') - \tilde{q}_{2j+1}^I\tau_{2j}^I(v')\right] \tag{4.1.31}$$

$$IS_N^I(v,v') = 2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^I}\left[\tau_{2j}^I(w)\tau_{2j+1}^I(w') - \tau_{2j+1}^I(w)\tau_{2j}^I(v')\right]. \tag{4.1.32}$$

Furthermore note that the eigenvalue densities for finite matrix dimensions N can be read off from the (K', L')-correlation functions in equation (4.1.20) specializing to the $(0,1)$ and $(1,0)$ cases. Indeed for the induced family of real asymmetric matrices:

$$\rho_{I,N}^{\mathbb{C}}(z) := R_{0,1}^I(-,z) = \text{Pfaff } K_N^I(z,z) = S_N^I(z,z) \quad (z \in \mathbb{C}_+)\ , \tag{4.1.33}$$

is the mean density of complex eigenvalues, whilst:

$$\rho_{I,N}^{\mathbb{R}}(x) := R_{1,0}^I(x,-) = \text{Pfaff } K_N^I(x,x) = S_N^I(x,x) \quad (x \in \mathbb{R}) \tag{4.1.34}$$

is the mean density of real eigenvalues. Note the normalization:

$$2\int_{\mathbb{C}_+} \rho_{I,N}^{\mathbb{C}}(z)\ d^2z + \int_{\mathbb{R}} \rho_{I,N}^{\mathbb{R}}(x)\ dx = N\ . \tag{4.1.35}$$

4.1.3 The characteristic average

The direct computation of skew-orthogonal polynomials with respect to an inner skew-product $(-,-)$ is a tremendous task for almost all weight functions. As a result a different approach is employed in order to determine the required skew-orthogonal polynomials and thus the kernel entries in theorem 4.1.8. We show that it is possible to relate the mean density of complex eigenvalues to an average of the characteristic polynomial over the respective matrix measure. The mean density of complex eigenvalues is related to the kernel entry S_N, as can be seen in equation (4.1.33) and thus a connection between the Pfaffian kernel entries from equations (4.1.30) - (4.1.32) and the characteristic average can be made. This connection is further on exploited in order to explicitly determine the necessary skew-orthogonal polynomials.

As already observed in [Som07, APS09b] for the real Ginibre ensemble the following relationship between the complex mean eigenvalue density and the characteristic average holds true:

Theorem 4.1.10. *The mean density of complex eigenvalues $\rho_N^{\mathbb{C}}(z)$ of a real asymmetric matrix $A_N \in \mathbb{R}^{N\times N}$ with matrix measure $\mathrm{d}\,\mu_{I,1}^{Induced} = P_{I,1}^{Induced}(A_N)(dA_N)$ for $I \in \{Ginibre, Jacobi, Spherical\}$ can be related to the characteristic average of the matrix $A_{N-2} \in \mathbb{R}^{N-2\times N-2}$ with measure $\mathrm{d}\,\mu_{I,1}^{Induced} = P_{I,1}^{Induced}(A_{N-2})(dA_{N-2})$ as follows:*

$$\rho_{I,N}^{\mathbb{C}}(z) = \frac{c_{N,k,l}^{I}}{c_{N-2,k,l-1}^{I}}(z-\bar{z})\left\langle \det\left((A_{N-2}-zI_{N-2})(A_{N-2}-\bar{z}I_{N-2})\right)\right\rangle_{A_{N-2}}^{I}. \quad (4.1.36)$$

where $c_{N,k,l}^{I}$ denotes the normalization constant of the respective eigenvalue jpdf and $\langle\rangle_{A_{N-2}}^{I}$ denotes the average with respect to the measure $\mathrm{d}\,\mu_{I,1}^{Induced}$.

Proof. We start by rewriting the mean eigenvalue density using definition 4.1.7 of the (K', L')-correlation functions:

$$\begin{aligned}\rho_{I,N}^{\mathbb{C}}(z) &= \sum_{l'=1}^{\frac{N}{2}} R_{(0,1,k',l')}^{I}(z)\\ &= \sum_{l'=1}^{\frac{N}{2}} l'!2^{l'-1}\int_{\mathbb{R}^{k'}}\int_{\mathbb{C}_+^{2(l'-1)}} P_{N,k',l'}^{I}(\lambda_1,\ldots,\lambda_{k'},z,z_2,\ldots,z_{l'})d\lambda_1\cdots d\lambda_{k'}d^2z_2\cdots d^2z_{l'}\,,\end{aligned}$$

and we used definition 4.1.6 of the partial (K', L', k', l')-correlation functions.

$$\begin{aligned}\rho_{I,N}^{\mathbb{C}}(z) = c_{N,k,l}^{I}&(z-\bar{z})w_{I,1}(z)w_{I,1}(\bar{z})\times\\ &\int_{\mathbb{R}^{k'}}\int_{\mathbb{C}_+^{2(l'-1)}}\sum_{l'=1}^{\frac{N}{2}} l'!2^{l'-1}\prod_{i_1<i_2}(\lambda_{i_1}-\lambda_{i_2})\prod_{i=1}^{k'} w_{I,1}(\lambda_i)(z-\lambda_i)(\bar{z}-\lambda_i)\\ &\prod_{j_1<j_2}(z_{j_1}-z_{j_2})\prod_{j=2}^{l'} w_{I,1}(z_j)(z-z_j)(\bar{z}-z_j)d\lambda_1\cdots d\lambda_{k'}d^2z_2\cdots d^2z_{l'}.\end{aligned} \quad (4.1.37)$$

Note that inside the integral is the eigenvalue jpdf of a $N-2\times N-2$ matrix with matrix measure $\mathrm{d}\,\mu_I$ with k' real eigenvalues and $l'-1$ complex conjugated eigenvalue pairs. As a result we can write:

$$\rho^{\mathbb{C}}_{I,N}(z) = \frac{c^I_{N,k,l}}{c^I_{N-2,k,l-1}}(z-\bar{z})w_{I,1}(z)w_{I,1}(\bar{z}) \times$$
$$\Big\langle \sum_{k'=0}^{N-2} l'!2^{l'-1} \prod_{i=1}^{k'}(z-\lambda_i)(\bar{z}-\lambda_i) \prod_{j=2}^{l'}(z-z_j)(\bar{z}-z_j) \Big\rangle^I_{A_{N-2}}. \qquad (4.1.38)$$

The expression inside the average is nothing but the characteristic polynomial of A_{N-2}, which proves our result. $\square$

A consequence of equation (4.1.36) is the corollary 4.1.11 below.

Corollary 4.1.11. *The kernel entry DS^I_N from 4.1.20 can be related to the characteristic average as follows:*

$$2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r^I_j}\left[q^I_{2j}(v)q^I_{2j+1}(v') - q^I_{2j+1}(v)q^I_{2j}(v')\right] = \frac{1}{r^I_N}\times$$
$$(v-v')\langle \det(G-vI)\det(G-v'I)\rangle^I_{G_{N-2}}, \qquad (4.1.39)$$

where $\langle \ldots \rangle^I_{G_{N-2}}$ denotes the average with respect to the matrix measure $d\mu^{Induced}_{I,1}$ and r^I_N is the normalization of the $N-$th skew-orthogonal polynomial as defined in (4.1.28).

Proof. Note that the mean density of complex eigenvalues is obtained from the kernel entry S_N, as seen in equation (4.1.33). In addition note that, for all three real induced ensembles the complex density of mean eigenvalues is analytic. It can be understood as an analytic function in the variables z and $\bar{z}$. As a result equation (4.1.36) can be extended to:

$$S(v,v') = \frac{c^I_{N,k,l}}{c^I_{N-2,k,l-1}}(v-v')\Big\langle \det\big((A-vI_{N-2})(A-v'I_{N-2})\big)\Big\rangle^I_{A_{N-2}}. \qquad (4.1.40)$$

The corollary now follows from the definition of the kernel entry S_N, see equation (4.1.31). $\square$

Instead of explicitly computing the respective skew-orthogonal polynomials it is possible to obtain the kernel entries of the Pfaffian correlation kernel by calculating the characteristic average. Furthermore it transpires, that the characteristic average can be computed with the help of elementary symmetric functions:

Theorem 4.1.12. *[[KSŻ10, WK09]] Let $A = \mathrm{diag}(a_1, \ldots, a_n)$ and let $\epsilon_j(AA^T)$ be the j-th symmetric polynomial in the eigenvalues $l_1, \ldots, l_N$ of AA^T:*

$$\epsilon_j(AA^T) := \epsilon_j(l_1, \ldots, l_N) = \sum_{1 \leq i_1 < \cdots < i_j \leq N} l_{i_1} \cdots l_{i_N}. \tag{4.1.41}$$

In addition let $Q \in O(N)$. Then:

$$\left\langle \left| \det\left(AQ + zI_N\right) \right|^2 \right\rangle_{O(N)} = \sum_{j=0}^{N} \frac{\epsilon_j(AA^T)}{\epsilon_j(I_N)} |z|^{2(N-j)}. \tag{4.1.42}$$

Remark 4.1.13. *Surprisingly the above identity also holds for $U(N)$*

$$\left\langle \left| \det\left(AQ + zI_N\right) \right|^2 \right\rangle_{U(N)} = \sum_{j=0}^{N} \frac{\epsilon_j(AA^\dagger)}{\epsilon_j(I_N)} |z|^{2(N-j)}. \tag{4.1.43}$$

Remark 4.1.14. *Note that*

$$\epsilon_j(I_N) = \binom{N}{j} \tag{4.1.44}$$

Proof. The first step is to expand the characteristic polynomials in terms of elementary symmetric polynomials in the eigenvalues of AQ:

$$\det\left(AQ + zI_N\right) = \sum_{j=0}^{N} z^{N-j} \epsilon_j(AQ), \tag{4.1.45}$$

which immediately yields:

$$\left\langle \left| \det\left(AQ + zI_N\right) \right|^2 \right\rangle_{O(N)} = \sum_{j,j'=0}^{N} z^{N-j} \bar{z}^{N-j} \left\langle \epsilon_j(AQ) \epsilon_{j'}(AQ) \right\rangle_{O(N)}. \tag{4.1.46}$$

The average over the cross-product of elementary symmetric functions can be dealt with by expanding $\epsilon_j(AQ)$ in the principal minors of AQ:

$$\epsilon_j(AQ) = \sum_{1 \leq i_1 < \cdots < i_j \leq N} l_{i_1} \cdots l_{i_j} \det\left[\left(Q_{i_m, i_n}\right)_{m,n=1}^{j}\right]. \tag{4.1.47}$$

Now note

1. The average of the product of two principal minors of Q is zero, unless the minors are identical:

$$\left\langle \det\left[\left(Q_{i_m, i_n}\right)_{m,n=1}^{j}\right] \det\left[\left(Q_{i'_m, i'_n}\right)_{m,n=1}^{j'}\right] \right\rangle_{O(N)} = \gamma_{j,j'} \delta_{j,j'} \delta_{i_1, i'_1} \cdots \delta_{i_j, i'_j}.$$

This is due to the invariance of the Haar measure with respect to left and right translation. If two minors of Q are not identical, one can always find a row that is present in one but not the other. One can always change the sign of that row by multiplying Q to the left by an appropriate diagonal matrix of plus and minus ones. Since the average is invariant such a transformation should not change the value of the average. But this is only possible, if the average is zero.

2. The average of the square of a principal minor of Q does not depend on the choice of columns:

$$\left\langle \left[\det \left(Q_{i_m,i_n}\right)_{m,n=1}^{j} \right]^2 \right\rangle_{O(U)} = \left\langle \left[\det \left(Q_{m,n}\right)_{m,n=1}^{j} \right]^2 \right\rangle_{O(U)}.$$

This follows again from the invariance of the Haar measure as it is possible to swap any rows or columns of Q by multiplying Q either to the left or right by an appropriate elementary permutation matrix. Thus one can reduce any principal minor of Q to the top left right block of Q through a similarity transformation: $P^T Q P$ with $P \in O(N)$.

Consequently the following orthogonality relation holds:

$$\left\langle \epsilon_j(AQ)\epsilon_{j'}(AQ)\right\rangle_{O(N)} = \delta_{j,j'} \left\langle \left[\det(Q_{m,n})_{m,n=1}^{j} \right]^2 \right\rangle_{O(N)} \epsilon_j(AA^T). \tag{4.1.48}$$

It thus remains to calculate the coefficients:

$$\left\langle \left[\det(Q_{m,n})_{m,n=1}^{j} \right]^2 \right\rangle_{O(N)}. \tag{4.1.49}$$

The coefficient can be calculated from the generating function:

$$F(x) = \langle \det(xT+Q)^2 \rangle_{O(N)} = \sum_{j=0}^{N} x^{2(N-j)} \binom{n}{j} \left\langle \left[\det(Q_{m,n})_{m,n=1}^{j} \right]^2 \right\rangle_{O(N)}. \tag{4.1.50}$$

Furthermore the idea is to expand $\det \left(xI + Q\right)^2$ in Schur functions using the dual Cauchy identity, page 63-65 [Mac95] as follows:

$$\det \left(xI + Q\right)^2 = \sum_{\lambda} x^{|\lambda'|} s_{\lambda'}(1,1) s_{\lambda}(z_1, \ldots, z_N), \tag{4.1.51}$$

where $z_1, \ldots, z_N$ are the eigenvalues of Q and the summation is over partitions

$\lambda = (\lambda_1, \ldots, \lambda_N)$, $2 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$. Hence:

$$\big\langle (\det\big(xI + Q\big)^2 \big\rangle_{O(N)} = \sum_{\lambda} x^{|\lambda'|} s_{\lambda'}(I_2) \langle s_\lambda(Q) \rangle_{O(N)}. \tag{4.1.52}$$

We can use:

Theorem 4.1.15 ([Mac95], page 420-421)**.** *The expression* $\big\langle s_\lambda(Q) \big\rangle_{O(N)}$ *vanishes unless* λ *is an even partition and* $\big\langle s_{2\lambda}(Q) \big\rangle_{O(N)} = 1$*.*

Our partitions have at most two columns in their Young diagram, therefore the only non-zero terms in expansion 4.1.52 will be those corresponding to rectangles $r \times 2$, $r = 1, \ldots, N$ and the empty partition. If $\lambda = (2, \ldots, 2)$ then $\lambda' = (r, r)$ and $s_{(r,r)}(I_2)$=1. Consequently:

$$\big\langle (\det\big(xI + Q\big)^2 \big\rangle_{O(N)} = \sum_{r=0}^{N} x^{2r}, \tag{4.1.53}$$

which in turn implies:

$$\big\langle \epsilon_j(AQ) \epsilon_{j'}(AQ) \big\rangle_{O(N)} = \delta_{j,j'} \frac{\epsilon_j(AA^T)}{\epsilon_j(I)}. \tag{4.1.54}$$

□

See [FK07] for a similar integral over the unitary group. Applying theorem 4.1.12 the characteristic average over the real induced Ginibre ensemble can easily be determined.

Theorem 4.1.16. *In the case of the real induced Ginibre ensemble specified by the matrix measure* $\mathrm{d}\,\mu^{\textit{Induced}}_{\textit{Ginibre,1}}$*:*

$$\Big\langle \det\big(A - zI_m\big) \det\big(A - vI_m\big) \Big\rangle^{\textit{IndGin}}_{A_m} = \Gamma(L + m + 1) \sum_{j=0}^{m} \frac{(zv)^j}{\Gamma(L + j + 1)}. \tag{4.1.55}$$

Proof. The proof depends on the fact, that the average $\langle \rangle^{\text{IndGin}}_{A_m}$ is invariant with respect to orthogonal transformation. This is due to the determinant and probability measure being invariant with respect to orthogonal transformation. We may then write for an orthogonal matrix $Q \in O(N)$:

$$\Big\langle \det\big((A - zI_m)(A - vI_m)\big) \Big\rangle^{\text{IndGin}}_{A_m} = \Big\langle \Big\langle \det\big((AQ - zI_m)(AQ - vI_m)\big) \Big\rangle_{O(N)} \Big\rangle^{\text{IndGin}}_{A_m}.$$

Consequently we can apply theorem 4.1.12 and obtain:

$$\begin{aligned}
&\Big\langle \det\big((A_m - zI_m)(A_m - vI_m)\big)\Big\rangle_{A_m}^{\mathrm{IndGin}} \\
=&\Big\langle \sum_{j=0}^{m} \frac{\epsilon_j(A_m A_m^T)}{\binom{m}{j}}(zv)^{m-j}\Big\rangle_{A_m}^{\mathrm{IndGin}} = \sum_{j=0}^{m} \frac{\langle \epsilon_j(A_m A_m^T)\rangle_{A_m}^{\mathrm{IndGin}}}{\binom{m}{j}}(zv)^{m-j}.
\end{aligned} \tag{4.1.56}$$

The average:

$$\begin{aligned}
&\langle \epsilon_j(A_m A_m^T)\rangle_{A_m}^{\mathrm{IndGin}} \\
=&c_{N,k,l}\int_{(A_m)} \sum_{1\le i_1<\cdots<i_j\le m} l_{i_1}\cdots l_{i_j} \det(A_m A_m^T)^{\frac{L}{2}} e^{-\frac{1}{2}\operatorname{tr}(A_m A_m^T)}(dA_m)
\end{aligned} \tag{4.1.57}$$

can be reduced to a Selberg-Aomoto integral [Meh04], see also theorem D.1.1. First change variables to the singular value decomposition $A_m = U\Sigma V$ with $\sigma_1,\ldots,\sigma_m$ the singular values of A with Jacobian $\prod_{i<j}|\sigma_i^2-\sigma_j^2|$ and note that $\sigma_j^2 = l_j$, for $j=1,\ldots,m$. In addition we note that the expressions are symmetric in the eigenvalues. Each term in the sum of eigenvalues is of length j and all terms are distinct. Thus there are $\binom{m}{j}$ terms in the sum. Moreover we remove the ordering of the singular values which gives a factor of $m!$. As a result:

$$\begin{aligned}
&\langle \epsilon_j(A_m A_m^T)\rangle_{A_m} = c_{N,k,l}\binom{m}{j} m! \times \\
&\int_{(\Sigma)}\int_{O(m)}\int_{O[m]} \prod_{i_1<i_2}|\sigma_{i_1}^2-\sigma_{i_2}^2| \prod_{i=1}^{j}\sigma_i^2 \det(\Sigma)^L \,\mathrm{e}^{-\frac{1}{2}\Sigma^2}(d\Sigma)(U^T dU)(V^T dV) \\
=&c_{N,k,l}\binom{m}{j} m! \operatorname{Vol}(O(m))\operatorname{Vol}(O[m])2^{-m} \\
&\int_0^\infty\cdots\int_0^\infty \prod_{i_1<i_2}|s_{i_1}-s_{i_2}| \prod_{i=1}^{j} s_i \prod_{i=1}^{m} s_i^L\,\mathrm{e}^{-\frac{1}{2}s_i}\,ds_1\cdots ds_m.
\end{aligned} \tag{4.1.58}$$

Using theorem D.1.1 then gives the desired result. $\square$

4.1.4 The (K', L')-correlation functions and mean eigenvalue densities

A major point of this work, albeit not a new idea, is remarking, that equation (4.1.11) can be exploited to determine the skew-orthogonal polynomials needed for the derivation of the Pfaffian kernel entries. In the context of the real induced

Ginibre ensemble corollary 4.1.11 gives:

$$
\begin{aligned}
&2\sum_{j=0}^{\frac{N}{2}-1}\frac{1}{r_j^{\text{IndGin}}}\left[q_{2j}^{\text{IndGin}}(w)q_{2j+1}^{\text{IndGin}}(w')-q_{2j+1}^{\text{IndGin}}(w)q_{2j}^{\text{IndGin}}(w')\right]\\
=&\frac{1}{r_N^{\text{IndGin}}}(w-w')\frac{\Gamma(L+N-1)}{\sqrt{2\pi}}\sum_{j=0}^{N-2}\frac{(ww')^j}{\Gamma(L+j+1)}\ . \qquad (4.1.59)
\end{aligned}
$$

As already observed in [APS09b] the skew-orthogonal polynomials can now be just "read off" using the fact that each q_j^{IndGin} is monic and of degree j by, for example, differentiating:

$$
\begin{aligned}
q_{2j}^{\text{IndGin}}(w) &= r_j^{\text{IndGin}}\tfrac{1}{(2j+1)!}\tfrac{\partial^{2j+1}}{\partial u^{2j+1}}\left[\tfrac{\Gamma(L+N-1)}{r_N^{\text{IndGin}}}(u-w)\textstyle\sum_{j=0}^{2j}\tfrac{(wu)^j}{\Gamma(L+j+1)}\right]\Big|_{u=0}\\
q_{2j+1}^{\text{IndGin}}(w) &= r_j^{\text{IndGin}}\tfrac{1}{(2j)!}\tfrac{\partial^{2j}}{\partial u^{2j}}\left[\tfrac{\Gamma(L+N-1)}{r_N^{\text{IndGin}}}(w-u)\textstyle\sum_{j=0}^{2j}\tfrac{(wu)^j}{\Gamma(L+j+1)}\right]\Big|_{u=0}.
\end{aligned}
$$

Hence,

Theorem 4.1.17. *For $j=1,2,\ldots$ the following polynomials were found to be skew-orthogonal with respect to the skew-inner product $(-,-)^{IndGin}$:*

$$q_{2j}^{IndGin}(z)=z^{2j},\qquad q_{2j+1}^{IndGin}(z)=z^{2j+1}-(2j+L)z^{2j-1}. \qquad (4.1.60)$$

In addition the first two skew-orthogonal polynomials are given by: $q_0^{IndGin}(z)=1$ and $q_1^{IndGin}(z)=z$. The normalization constant is given by:

$$r_j^{IndGin}=(q_{2j}^{IndGin},q_{2j+1}^{IndGin})_{IndGin}=2\sqrt{2\pi}\,\Gamma(L+2j+1). \qquad (4.1.61)$$

Thus the entries of the Pfaffian kernel can be now be explicitly determined. It is convenient to define:

$$
\begin{aligned}
t^{\text{IndGin}}(x,z) &= \frac{1}{\sqrt{2\pi}}w_{\text{IndGin},1}(z)2^{\frac{L}{2}-1}\frac{\Gamma(\frac{L}{2},\frac{1}{2}x^2)}{\Gamma(L)}; \qquad (4.1.62)\\
s_N^{\text{IndGin}}(z,v) &= \frac{1}{\sqrt{2\pi}}w_{\text{IndGin},1}(z)w_{\text{IndGin},1}(v)\sum_{j=0}^{N-2}\frac{(vz)^j}{\Gamma(L+j+1)}; \qquad (4.1.63)\\
r_N^{\text{IndGin}}(x,z) &= \frac{1}{\sqrt{2\pi}}w_{\text{IndGin},1}(z)\,\text{sgn}(x)2^{\frac{N}{2}+\frac{L}{2}-\frac{3}{2}}z^{N-1}\frac{\gamma(\frac{N}{2}+\frac{L}{2}-\frac{1}{2},\frac{1}{2}x^2)}{\Gamma(N+L-1)}. \qquad (4.1.64)
\end{aligned}
$$

Moreover note that $s_N^{\text{IndGin}}(z,v)$ is symmetric in its variables and $t^{\text{IndGin}}(x,z)$ and $r_N^{\text{IndGin}}(x,z)$ are not.

Theorem 4.1.18. *For the real induced Ginibre ensemble the entries of the complex/complex (2×2) matrix kernel $K_N^{IndGin}(z,v)$ in (4.1.20)–(4.1.21) are given* *by:*

$$DS_N^{IndGin}(z,v) = (v-z)s_N^{IndGin}(z,v); \tag{4.1.65}$$
$$S_N^{IndGin}(z,v) = i(\bar{v}-z)s_N^{IndGin}(z,\bar{v}); \tag{4.1.66}$$
$$IS_N^{IndGin}(z,v) = (\bar{z}-\bar{v})s_N^{IndGin}(\bar{z},\bar{v}). \tag{4.1.67}$$

The entries of the real/complex and complex/real matrix kernels $K_N^{IndGin}(x,z)$ *and* $K_N^{IndGin}(z,x)$ *in (4.1.30)–(4.1.32) are given by:*

$$DS_N^{IndGin}(x,z) = (z-x)s_N^{IndGin}(x,z); \tag{4.1.68}$$
$$DS_N^{IndGin}(z,x) = -DS_N^{IndGin}(x,z); \tag{4.1.69}$$
$$S_N^{IndGin}(x,z) = i(\bar{z}-x)s_N^{IndGin}(x,\bar{z}); \tag{4.1.70}$$
$$S_N^{IndGin}(z,x) = s_N^{IndGin}(x,z) + r_N^{IndGin}(x,z) + t^{IndGin}(x,z); \tag{4.1.71}$$
$$IS_N^{IndGin}(x,z) = -is_N^{IndGin}(x,\bar{z}) - ir_N^{IndGin}(x,\bar{z}) - it^{IndGin}(x,\bar{z}); \tag{4.1.72}$$
$$IS_N^{IndGin}(z,x) = -IS_N^{IndGin}(x,z)\ . \tag{4.1.73}$$

And finally, the entries of the real/real matrix kernel $K_N^{IndGin}(x,y)$ *in (4.1.20)–(4.1.30) are given by:*

$$DS_N^{IndGin}(x,y) = (y-x)s_N^{IndGin}(x,y); \tag{4.1.74}$$
$$S_N^{IndGin}(x,y) = s_N^{IndGin}(x,y) + r_N^{IndGin}(y,x) + t^{IndGin}(x,y); \tag{4.1.75}$$
$$\begin{aligned} IS_N^{IndGin}(x,y) =& \frac{1}{\sqrt{2\pi}}\Big[-\frac{\gamma(L,y^2)}{\Gamma(L)} + e^{-\frac{1}{2}(x-y)^2}\frac{\gamma(L,xy)}{\Gamma(L)} + \frac{y^L e^{\frac{1}{2}y^2}}{\Gamma(L)}\int_x^y e^{-\frac{1}{2}t^2}t^{L-1}dt \\ & + \frac{\gamma(L+N-1,y^2)}{L+N-1} - e^{-\frac{1}{2}(x-y)^2}\frac{\gamma(L+N-1,xy)}{\Gamma(L+N-1)} \\ & - \frac{y^{L+N-1}e^{\frac{1}{2}y^2}}{\Gamma(L+N-1)}\int_x^y e^{-\frac{1}{2}t^2}t^{L+N-2}dt \\ & - \mathrm{sgn}(y)2^{\frac{L}{2}+N-\frac{3}{2}}\frac{\gamma(\frac{L}{2}+\frac{N}{2}-\frac{1}{2},\frac{1}{2}y^2)}{\Gamma(L+N-1)}\int_x^y e^{-\frac{1}{2}t^2}t^{L+N-1}dt \\ & - 2^{\frac{L}{2}-1}\frac{\Gamma(\frac{L}{2},\frac{1}{2}y^2)}{\Gamma(L)}\int_x^y e^{-\frac{1}{2}t^2}t^L dt\Big]. \end{aligned} \tag{4.1.76}$$

The proof of theorem 4.1.18 is delegated to the appendix, section C.1. Theorem 4.1.18 now yields the finite-N complex and real eigenvalue densities for the real induced Ginibre ensemble in a closed form:

$$\begin{aligned}\rho^{\mathbb{C}}_{\mathrm{IndGin},N}(x+iy) &= \sqrt{\frac{2}{\pi}}y\,\mathrm{erfc}(\sqrt{2}y)e^{y^2-x^2}\sum_{j=0}^{N-2}\frac{(x^2+y^2)^{j+L}}{\Gamma(j+L+1)} \\ &= \sqrt{\frac{2}{\pi}}y\,\mathrm{erfc}(\sqrt{2}y)e^{2y^2}\left[\frac{\gamma(L,x^2+y^2)}{\Gamma(L)}-\frac{\gamma(L+N-1,x^2+y^2)}{\Gamma(L+N-1)}\right],\end{aligned} \tag{4.1.77}$$

and

$$\begin{aligned}\rho^{\mathbb{R}}_{\mathrm{IndGin},N}(x) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2}\sum_{j=0}^{N-2}\frac{x^{2(j+L)}}{\Gamma(j+L+1)}+t^{\mathrm{IndGin}}(x,x)+r_N^{\mathrm{IndGin}}(x,x) \\ &= \frac{1}{\sqrt{2\pi}}\Big[\frac{\gamma(L,x^2)}{\Gamma(L)}-\frac{\gamma(L+N-1,x^2)}{\Gamma(N+L-1)}\Big]+t^{\mathrm{IndGin}}(x,x)+r_N^{\mathrm{IndGin}}(x,x)\,.\end{aligned} \tag{4.1.78}$$

4.1.5 Asymptotic analysis

Exactly like in the case of the complex induced Ginibre ensemble in the limit of large matrix dimensions it is possible to distinguish two asymptotic regimes: the regime of strong rectangularity and the regime of almost square matrices. In the regime of strong rectangularity the eigenvalue repulsion from the origin is strong and as a result the mean density of complex eigenvalues is to leading order uniform on an annulus, whose width depends on the rectangularity parameter L. Furthermore on the support of the eigenvalue density in the bulk and at the edge of the support, the correlation kernel show universal behavior, again meaning that in the limit of large matrix dimension the limiting kernels of the complex Ginibre ensemble are recovered.

In the regime of almost square matrices the parameter L is kept fixed and thus the mismatch in dimensions is kept small. As a result the repulsion away from the origin is weak and only creates a microscopically small hole. As a consequence the scaled eigenvalues are (to leading order) uniformly distributed on the unit disk. In the bulk and at the edge of the eigenvalue support we can again show that the correlation kernels exhibit universal behavior. As in the complex induced Ginibre ensemble, one of the main results of this work is, that at the origin a new correlation kernel emerges in the limit of large matrix dimensions. Indeed it seems that this correlation kernel is universal, in the sense that it can be recovered in different asymptotic regimes for the two additional ensembles studied in this work.

Figure 4.1 shows the eigenvalue distribution of the real induced Ginibre ensemble in the two asymptotic regimes.

Strong rectangularity

In this section we shall investigate the real induced Ginibre ensemble in the scaling limit when the free parameter L grows proportionally with the matrix dimension N, which, in the language of quadratization of rectangular matrices, corresponds to tall rectangular matrices which are neither skinny nor almost-square.

In the leading order, the distribution of complex eigenvalues turns out to be uniform in an annulus with the inner and outer radii $r_{in} = \sqrt{L}$ and $r_{out} = \sqrt{L+N}$, exactly as in the complex induced Ginibre ensemble.

Similarly, the saddle-point analysis of each of the incomplete Gamma functions in equation (4.1.78) yields the limiting density of real eigenvalues. In the leading order, the real eigenvalues in the induced Ginibre ensemble populate two symmetric segments of the real axis, $[r_{in}, r_{out}]$ and $[-r_{out}, -r_{in}]$, with constant density. The theorem below summarizes our findings.

Theorem 4.1.19. *Suppose that $L = N\alpha$ with $\alpha > 0$. Then:*

(a) In the leading order as $N \to \infty$, the average number of real eigenvalues in the real induced Ginibre ensemble is $\sqrt{\frac{2}{\pi}}(\sqrt{L+N} - \sqrt{L})$ and the density of real eigenvalues obeys the following limiting relation:

$$\lim_{N\to\infty} \rho^{\mathbb{R}}_{IndGin,N}(\sqrt{N}x) = \frac{1}{\sqrt{2\pi}} \left[\Theta(|x| - \sqrt{\alpha}) - \Theta(|x| - \sqrt{\alpha+1})\right] . \quad (4.1.79)$$

(b) The density of complex eigenvalues obeys the following limiting relation:

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{IndGin,N}(\sqrt{N}z) = \frac{1}{\pi} \left[\Theta(|z| - \sqrt{\alpha}) - \Theta(|z| - \sqrt{\alpha+1})\right] . \quad (4.1.80)$$

Proof. We start with the mean density of complex eigenvalues:

$$\rho^{\mathbb{C}}_{\mathrm{IndGin},N}(\sqrt{N}z) = \sqrt{\frac{2N}{\pi}} \operatorname{Im}(z) \operatorname{erfc}(\sqrt{2N}\operatorname{Im}(z)) e^{2N\operatorname{Im}(z)^2} \times \left[\frac{\gamma(L, N|z|^2)}{\Gamma(L)} - \frac{\gamma(L+N-1, N|z|^2)}{\Gamma(L+N-1)}\right]$$

From [AS72], theorem 7.1.13 we know:

$$\sqrt{N}\operatorname{Im}(z)\operatorname{erfc}(\sqrt{2N}\operatorname{Im}(z))e^{2N\operatorname{Im}(z)^2} = (2\pi)^{-1} \quad (4.1.81)$$

combined with theorem A.1.1 this gives part (b) of our theorem. Furthermore the mean density of real eigenvalues is given by:

$$\begin{aligned}\rho^{\mathbb{R}}_{\text{IndGin},N}(\sqrt{N}x) =& \frac{1}{\sqrt{2\pi}}\Big[\frac{\gamma(L,Nx^2)}{\Gamma(L)} - \frac{\gamma(L+N-1,Nx^2)}{\Gamma(N+L-1)}\Big] \\ &+ t^{\text{IndGin}}(\sqrt{N}x,\sqrt{N}x) + r_N^{\text{IndGin}}(\sqrt{N}x,\sqrt{N}x) \ ,\end{aligned}$$

using theorem A.1.1 again and noting that:

$$\lim_{N\to\infty} t^{\text{IndGin}}(\sqrt{N}x,\sqrt{N}x) = \lim_{N\to\infty} r_N^{\text{IndGin}}(\sqrt{N}x,\sqrt{N}x) = 0, \tag{4.1.82}$$

then gives part (a) of our theorem. □

On setting $L = 0$ in the above results one recovers the expected number $\sqrt{2N/\pi}$ of real eigenvalues in the Ginibre ensemble [EKS94] together with the uniform densities of distribution of real and complex eigenvalues [EKS94, Ede97]. One can examine how quickly the eigenvalue density falls to zero when one moves away from the boundary of the eigenvalue support. At the inner and outer circular edges *away from the real line*, one recovers the same eigenvalue density profile as for the complex Ginibre ensemble [FH99].

Theorem 4.1.20. *Suppose that $L = N\alpha$ with $\alpha > 0$. Then, for fixed $\xi \in \mathbb{R}$ and $\phi \neq 0, \pi$:*

$$\begin{aligned}&\lim_{N\to\infty} \rho^{\mathbb{C}}_{\textit{IndGin},N}((\sqrt{L}-\xi)e^{i\phi}) \\ &= \lim_{N\to\infty} \rho^{\mathbb{C}}_{\textit{IndGin},N}((\sqrt{L+N}+\xi)e^{i\phi}) = \frac{1}{2\pi}\operatorname{erfc}(\sqrt{2}\xi).\end{aligned} \tag{4.1.83}$$

The proof of theorem 4.1.20 is straightforward application of theorem A.1.2 and thus will be omitted. It follows from equation (4.1.83) that the density of complex eigenvalues in the real induced Ginibre ensemble falls to zero very fast (at a Gaussian rate) away from the boundary of the eigenvalue support as in the case of the complex induced Ginibre ensemble. Additionally for the density of real eigenvalues:

Theorem 4.1.21. *Suppose that $L = N\alpha$ with $\alpha > 0$. Then, for fixed $\xi \in \mathbb{R}$,*

$$\begin{aligned}&\lim_{N\to\infty} \rho^{\mathbb{R}}_{\textit{IndGin},N}(\sqrt{L}-\xi) \\ &= \lim_{N\to\infty} \rho^{\mathbb{R}}_{\textit{IndGin},N}(\sqrt{L+N}+\xi) = \frac{1}{\sqrt{2\pi}}\left[\operatorname{erfc}(\sqrt{2}\xi) + \frac{1}{2\sqrt{2}}e^{-\xi^2}\operatorname{erfc}(-\xi)\right].\end{aligned} \tag{4.1.84}$$

Proof. We start with the inner edge: $x = \sqrt{L}-\xi$. Then using theorem A.1.1

gives:

$$\lim_{N\to\infty} r_N^{\mathrm{IndGinibre}}(\sqrt{L}-\xi,\sqrt{L}-\xi) = 0. \tag{4.1.85}$$

In addition:

$$\begin{aligned}
&t^{\mathrm{IndGin}}(\sqrt{L}-\xi,\sqrt{L}-\xi)\\
=&\frac{1}{2\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}L+\sqrt{L}\xi-\frac{1}{2}\xi^2}\left(\frac{L}{2}\right)^{\frac{L}{2}}2^L\big(1-\frac{\xi}{\sqrt{L}}\big)^L\,\frac{\Gamma(\frac{1}{2}L,\frac{1}{2}L-\sqrt{L}\xi+\frac{1}{2}\xi^2)}{\Gamma(L)}\\
=&\frac{1}{\sqrt{2}}\,\mathrm{e}^{-\frac{1}{2}L+\sqrt{L}\xi-\frac{1}{2}\xi^2}\left(\frac{L}{2}\right)^{\frac{L}{2}}\big(1-\frac{\xi}{\sqrt{L}}\big)^L\,\frac{\Gamma(\frac{1}{2}L,\frac{1}{2}L-\sqrt{L}\xi+\frac{1}{2}\xi^2)}{\Gamma\big(\frac{L}{2}\big)\Gamma\big(\frac{L+1}{2}\big)},
\end{aligned} \tag{4.1.86}$$

where we used the gamma doubling formula. Furthermore $\big(1-\frac{\xi}{\sqrt{L}}\big)^L \sim \mathrm{e}^{-\sqrt{L}\xi-\frac{1}{2}\xi^2}$ and Stirling's formula gives: $\Gamma\big(\frac{L+1}{2}\big)\sim \mathrm{e}^{-\frac{L+1}{2}}\big(\frac{L+1}{2}\big)^{\frac{L+1}{2}}\sqrt{\frac{4\pi}{L+1}}$. All in all:

$$t^{\mathrm{IndGin}}(\sqrt{L}-\xi,\sqrt{L}-\xi)\sim\frac{1}{2\sqrt{2\pi}}\,\mathrm{e}^{-\xi^2}\,\frac{\Gamma(\frac{1}{2}L,\frac{1}{2}L-\sqrt{L}\xi+\frac{1}{2}\xi^2)}{\Gamma\big(\frac{L}{2}\big)}. \tag{4.1.87}$$

Applying theorem A.1.2 then yields:

$$t^{\mathrm{IndGin}}(\sqrt{L}-\xi,\sqrt{L}-\xi)\sim\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{2}}\,\mathrm{e}^{-\xi^2}\,\mathrm{erfc}\,(\xi). \tag{4.1.88}$$

At the outer edge $x=\sqrt{L+N}+\xi$ using A.1.1 gives:

$$\lim_{N\to\infty} t^{\mathrm{IndGin}}(\sqrt{L+N}+\xi,\sqrt{L+N}+\xi)=0, \tag{4.1.89}$$

while we can show as before that:

$$r_N^{\mathrm{IndGin}}(\sqrt{L+N}+\xi,\sqrt{L+N}+\xi)\sim\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{2}}\,\mathrm{e}^{-\xi^2}\,\mathrm{erfc}\,(\xi). \tag{4.1.90}$$

The result now follows by applying theorem A.1.2. □

Another interesting transitional region appears close to the real line. Here the density of complex eigenvalues is more sparse: for finite matrix dimensions $\rho_N^C(x+iy)\propto y$ for small values of y. One can easily obtain the complex eigenvalue density profile in the crossover from zero density on the real axis to the plateau of constant density far away from the real axis. For example, at the origin $\lim_{N\to\infty}\rho^{\mathbb{C}}_{\mathrm{IndGin,N}}(iv)=\sqrt{\frac{2}{\pi}}v\,\mathrm{erfc}(\sqrt{2}v)\,\mathrm{e}^{2v^2}$, and more generally

Theorem 4.1.22. *In the vicinity of the real line $z=\sqrt{N}u+iv$ with $v\neq 0$ and*

$\sqrt{\alpha} < |u| < \sqrt{\alpha+1}$ *the limiting density of complex eigenvalues becomes:*

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{IndGin,N}(\sqrt{N}u + iv) = \sqrt{\frac{2}{\pi}} v \,\mathrm{erfc}(\sqrt{2}v)\, \mathrm{e}^{2v^2} . \tag{4.1.91}$$

Furthermore closing down on the outer real edge: $z = \sqrt{N(\alpha+1)} + iv$*:*

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{IndGin,N}(\sqrt{N}(\alpha+1) + iv) = \frac{1}{\sqrt{2\pi}} v \,\mathrm{erfc}\left(\sqrt{2}v\right) \mathrm{e}^{2v^2} . \tag{4.1.92}$$

Proof. In the bulk close to the real line the complex density becomes

$$\begin{aligned}\rho^{\mathbb{C}}_{\mathrm{IndGin},N}(\sqrt{N}u + iv) =& \sqrt{\frac{2}{\pi}} v \,\mathrm{erfc}\left(\sqrt{2}v\right) e^{2v^2} \times \\ & \left[\frac{\gamma\big(L, Nu^2+v^2\big)}{\Gamma(L)} - \frac{\gamma\big(L+N, Nu^2+v^2\big)}{\Gamma(L+N)}\right] .\end{aligned} \tag{4.1.93}$$

Applying theorem A.1.1 then proves both parts of the result. □

It should be noted that apart from the support of the eigenvalue distribution which clearly depends on α, the limiting eigenvalue density profiles in various scaling regimes in the induced Ginibre ensemble are independent of α and coincide with those for the original Ginibre ensemble. This correspondence also extends to the eigenvalue correlation functions. The eigenvalue correlation functions in the induced Ginibre ensemble in the bulk and at the edges are given by the expressions obtained for the Ginibre ensemble [BS09], see also [FN07, Som07, FN08]. A detailed analysis of the limiting behavior of the eigenvalue correlations in the bulk and at the edge of the eigenvalue distribution is undertaken in the appendix, section B.1. Similar calculations lead to the conclusion that in the complex bulk and also at the edges the eigenvalue correlation functions in the real induced Ginibre are exactly the same as those in the real Ginibre ensemble.

4.1.6 Almost square matrices

Another interesting regime arises when the rectangularity index L is fixed instead of growing proportionally with matrix size as discussed in the last section. In the bulk, i.e. at a distance of order $\sqrt{N}$ from the origin one recovers uniform distribution of eigenvalues (real and complex) and Ginibre correlations, whereas in the vicinity of the origin new eigenvalue statistics arise. The eigenvalue densities can be obtained by extending the summation in (4.1.77), (4.1.78) to infinity .

This yields:

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{\mathrm{IndGin},N}(x+iy) = \sqrt{\frac{2}{\pi}} y \,\mathrm{erfc}(\sqrt{2}y)\, e^{2y^2} \,\frac{\gamma(L, x^2+y^2)}{\Gamma(L)}, \tag{4.1.94}$$

for the density of complex eigenvalues and

$$\lim_{N\to\infty} \rho^{\mathbb{R}}_{\mathrm{IndGin},N}(x) \;= \tfrac{1}{\sqrt{2\pi}}\left[\tfrac{\gamma(L,x^2)}{\Gamma(L)} + e^{-\frac{1}{2}x^2} x^L 2^{\frac{L}{2}-1} \tfrac{\Gamma\left(\frac{L}{2},\frac{1}{2}x^2\right)}{\Gamma(L)}\right] \tag{4.1.95}$$

for the density of real eigenvalues. As in the case of complex matrices the higher order correlation functions at the origin are non-universal:

Theorem 4.1.23. *1. The limiting real/real kernel is given by a* 2×2 *matrix:*

$$\begin{aligned} &K^{\mathit{IndGin}}_{\mathit{origin}}(r,r') \\ =&\frac{1}{\sqrt{2\pi}}\left[\begin{array}{cc} (r'-r)e^{-\frac{1}{2}(r-r')^2}\frac{\gamma(L,rr')}{\Gamma(L)} & e^{-\frac{1}{2}(r-r')^2}\frac{\gamma(L,rr')}{\Gamma(L)} + t(r,r') \\ -e^{-\frac{1}{2}(r-r')^2}\frac{\gamma(L,rr')}{\Gamma(L)} - t(r,r') & (*) \end{array}\right]. \end{aligned} \tag{4.1.96}$$

where

$$\begin{aligned} (*) =& -\frac{\gamma(L,r'^2)}{\Gamma(L)} + e^{-\frac{1}{2}(r-r')^2}\frac{\gamma(L,rr')}{\Gamma(L)} \\ &+ \Big(\frac{r'^L e^{\frac{1}{2}r'^2}}{\Gamma(L)} - 2^{\frac{L}{2}-1}\frac{\Gamma(\frac{L}{2},\frac{1}{2}r'^2)}{\Gamma(\frac{L}{2})}\Big)\int_x^y e^{\frac{1}{2}t}t^L dt \end{aligned} \tag{4.1.97}$$

2. The limiting complex/complex kernel is given by a 2×2 *matrix:*

$$\begin{aligned} K^{\mathit{IndGin}}_{\mathit{origin}}(z,z') =& \frac{1}{\sqrt{2\pi}}\sqrt{\mathrm{erfc}(\sqrt{2}\,\mathrm{Im}(z))\,\mathrm{erfc}(\sqrt{2}\,\mathrm{Im}(z'))} \times \\ &\left[\begin{array}{cc} (z-z')e^{-\frac{1}{2}(z-z')^2}\frac{\gamma(L,zz')}{\Gamma(L)} & i(\bar{z}-z')e^{-\frac{1}{2}(z-\bar{z'})^2}\frac{\gamma(L,z\bar{z'})}{\Gamma(L)} \\ i(z'-\bar{z})e^{-\frac{1}{2}(\bar{z}-z')^2}\frac{\gamma(L,\bar{z}z')}{\Gamma(L)} & (\bar{z}-\bar{z'})e^{-\frac{1}{2}(\bar{z}-\bar{z'})^2}\frac{\gamma(L,\bar{z}\bar{z'})}{\Gamma(L)} \end{array}\right]. \end{aligned} \tag{4.1.98}$$

3. The limiting real/complex kernel is given by a 2×2 *matrix:*

$$\begin{aligned} K^{\mathit{IndGin}}_{\mathit{origin}}(r,z) =& \frac{1}{\sqrt{2\pi}}\sqrt{\mathrm{erfc}(\sqrt{2}\,\mathrm{Im}(z))} \times \\ &\left[\begin{array}{cc} (z-r)e^{-\frac{1}{2}(r-z)^2}\frac{\gamma(L,rz)}{\Gamma(L)} & i(\bar{z}-r)e^{-\frac{1}{2}(r-\bar{z})^2}\frac{\gamma(L,r\bar{z})}{\Gamma(L)} \\ -e^{-\frac{1}{2}(r-z)^2}\frac{\gamma(L,r\bar{z})}{\Gamma(L)} & -i\,\mathrm{e}^{-\frac{1}{2}(r-\bar{z})^2}\frac{\gamma(L,r\bar{z})}{\Gamma(L)} - it(r,\bar{z}) \end{array}\right]. \end{aligned} \tag{4.1.99}$$

Nevertheless setting the reference points at a distance of $\sqrt{N}$ away from the origin then yields the universal Ginibre correlation functions.

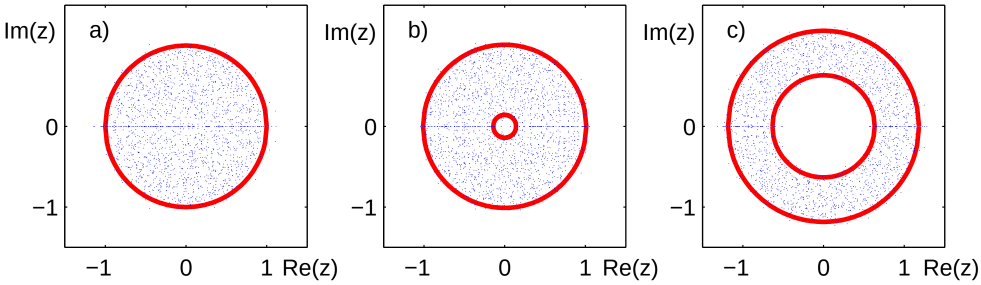


Figure 4.1: Spectra of matrices pertaining to the induced Ginibre ensemble of real matrices for dimension $N = 100$ and a) $L = 0$, b) $L = 2$, c) $L = 40$. Each plot consists of data from 20 independent realizations. The spectra are rescaled by a factor of $1/\sqrt{L+N}$ and the circles of radius $r_{\text{in}} = \sqrt{L/(L+N)}$ (inner one) and $r_{\text{out}} = 1$ (outer one) are depicted to guide the eye.

4.1.7 Summary of results

- The eigenvalue jpdf weight function of a real induced Ginibre matrix:

$$w_{\text{IndGin},1}(z) = z^L \, \mathrm{e}^{-\frac{1}{2}z^2} \left(\operatorname{erfc}\left(\sqrt{2}\,\operatorname{Im}(z)\right)\right)^{\frac{1}{2}}. \tag{4.1.100}$$

- The finite N mean density of complex eigenvalues for the real induced Ginibre ensemble:

$$\rho^{\mathbb{C}}_{\text{IndGin},N}(x+iy) = \sqrt{\frac{2}{\pi}} y \operatorname{erfc}(\sqrt{2}y) e^{2y^2} \times \left[\frac{\gamma(L, x^2+y^2)}{\Gamma(L)} - \frac{\gamma(L+N-1, x^2+y^2)}{\Gamma(L+N-1)}\right]. \tag{4.1.101}$$

- The finite N mean density of real eigenvalues for the real induced Ginibre ensemble:

$$\rho^{\mathbb{R}}_{\text{IndGin},N}(x) = \frac{1}{\sqrt{2\pi}} \left[\frac{\gamma(L, x^2)}{\Gamma(L)} - \frac{\gamma(L+N-1, x^2)}{\Gamma(N+L-1)}\right] + t^{\text{IndGin}}(x,x) + r_N^{\text{IndGin}}(x,x)\,. \tag{4.1.102}$$

- Limiting mean eigenvalue densities in the regime of strong rectangularity:

$$\lim_{N\to\infty} \rho^{\mathbb{R}}_{\mathrm{IndGin},N}(\sqrt{N}x) = \frac{1}{\sqrt{2\pi}}\left[\Theta(|x|-\sqrt{\alpha}) - \Theta(|x|-\sqrt{\alpha+1})\right] \quad (4.1.103)$$

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{\mathrm{IndGin},N}(\sqrt{N}z) = \frac{1}{\pi}\left[\Theta(|z|-\sqrt{\alpha}) - \Theta(|z|-\sqrt{\alpha+1})\right]. \quad (4.1.104)$$

- Limiting mean eigenvalue densities in the regime of almost square matrices:

$$\lim_{N\to\infty} \rho^{\mathbb{R}}_{\mathrm{IndGin},N}(\sqrt{N}x) = \frac{1}{\sqrt{2\pi}}\Theta(1-|x|) \quad (4.1.105)$$

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{\mathrm{IndGin},N}(\sqrt{N}z) = \frac{1}{\pi}\Theta(1-|z|). \quad (4.1.106)$$

- Limiting correlation kernel in the bulk in the regime of strong rectangularity: real Ginibre, see theorem B.1.1.

- Limiting correlation kernel in the bulk in the regime of almost square matrices: real Ginibre, see theorem B.1.1.
  Limiting correlation kernel at the origin in the regime of almost square matrices: new correlation kernel, see theorem 4.1.23.

4.2 The real induced Jacobi and the real induced spherical ensemble

In the following section the real counterparts of the induced spherical and induced Jacobi ensemble are introduced by applying the inducing procedure from chapter 2 to the rectangular real spherical ensemble and rectangular truncations of random orthogonal matrices. We then proceed to deriving the eigenvalue jpdfs for both ensembles. In addition applying the method of skew-orthogonal polynomials we derive the Pfaffian kernel entries of the (K', L')-correlation functions of both induced ensembles. As in the complex case an asymptotic analysis reveals four distinct asymptotic regimes for both ensembles.

4.2.1 The real induced spherical ensemble: Eigenvalue jpdf

The real spherical ensemble appears as early as the 1960's in the context of multivariate statistics as a multivariate generalization of the $t-$distribution in [Dic67]. It's eigenvalue distribution and correlation functions are studied in [FM11] in great detail. In the limit of large matrix dimensions the real spherical ensemble obeys the spherical law, meaning that in the limit of large matrix dimensions after an inverse stereographical projection, its eigenvalues are to leading order uniformly distributed on the unit sphere [Bor11].

Applying the inducing procedure to a matrix Y pertaining to the real rectangular spherical ensemble from definition 2.0.24 for $\beta = 1$, yields a random matrix A pertaining to the real induced spherical ensemble.

Definition 4.2.1. *The real induced spherical ensemble with parameters n, M is specified by the following probability measure on the space of $N \times N$ real matrices:* $\mathrm{d}\,\mu^{Induced}_{Spherical,1}(G) = P^{Induced}_{Spherical,1}(G)(dG)$,

$$P^{Induced}_{Spherical,1}(G) = C^{IndSpherical,1}_{M,N,n} \frac{\det(GG^T)^{\frac{M-N}{2}}}{\det(I_N + GG^T)^{\frac{n+M}{2}}} \quad M \geq N. \tag{4.2.1}$$

As in the previous sections we let $M - N := L$ denote the rectangularity parameter. Setting the parameters $n = M = N$ leads back to the real spherical ensemble from [FM11].

Lemma 4.2.2. *The element joint probability density function 4.2.1 is correctly normalized using:*

$$C^{IndSpherical,1}_{M,N,n} = \pi^{-\frac{1}{2}N^2} \prod_{j=1}^{N} \frac{\Gamma\big(\frac{j}{2}\big)\Gamma\big(\frac{n+L+j}{2}\big)}{\Gamma\big(\frac{L+j}{2}\big)\Gamma\big(\frac{n-N+j}{2}\big)}. \tag{4.2.2}$$

We can now use our knowledge of the element jpdf to derive the joint pdf for the eigenvalues of a real induced spherical random matrix. Again we need to assume that the matrix G pertaining to the real induced spherical ensemble has k real eigenvalues and l pairs of complex conjugated eigenvalues. The eigenvalue jpdf of the induced spherical ensemble is obtained by applying the method from [FM11].

Theorem 4.2.3. *The eigenvalue jpdf of a real induced spherical matrix with k real eigenvalues and l pairs of complex conjugated eigenvalues is given by:*

$$P^{IndSpherical}_{N,k,l}(\lambda_1, \ldots, \lambda_k, z_1, \ldots, z_l) = c^{IndSpherical}_{N,k,l} \big|\Delta\big(\{\lambda_j\}^k_{j=1} \cup \{z_m, \bar{z}_m\}^l_{m=1}\big)\big| \times$$
$$\prod_{j=1}^{k} w_{IndSpherical,1}(\lambda_j) \prod_{m=1}^{l} \mathrm{Im}(z_m) w_{IndSpherical,1}(z_m) w_{IndSpherical,1}(\bar{z}_m), \tag{4.2.3}$$

where

$$w_{IndSpherical,1}(z) = \frac{z^L}{|1+z^2|^{\frac{n+L+1}{2}}}\Big(\int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty}(1+u^2)^{-\frac{n+L+2}{2}}du\Big)^{\frac{1}{2}} \tag{4.2.4}$$

$$c_{N,k,l}^{IndSpherical} = 2^{3l}\pi^{-\frac{1}{2}l}\Big(\frac{\Gamma\big(\frac{n+L+1}{2}\big)}{\Gamma\big(\frac{n+L+2}{2}\big)}\Big)^{\frac{k}{2}}\prod_{j=1}^{N}\frac{\Gamma\big(\frac{n+L+1}{2}\big)^{\frac{1}{2}}\Gamma\big(\frac{n+L+2}{2}\big)^{\frac{1}{2}}}{\Gamma\big(\frac{j+L}{2}\big)\Gamma\big(\frac{n-N}{2}\big)} \tag{4.2.5}$$

as well as $\lambda_j \in \mathbb{R}_+$ *for* $j = 1,\dots,k$ *and* $z_m \in \mathbb{C}_+$ *for* $m = 1,\dots,l$. *Integrating the partial eigenvalue jpdf* $P_{N,k,l}^{IndSpherical}$ *over* $\mathbb{R}_+^k \times \mathbb{C}_+^{2l}$ *gives* $p_{N,k}^{IndSpherical}$, *the probability that a real induced spherical matrix of size* N *has* k *real eigenvalues and* l *pairs of complex conjugated eigenvalues.*

Proof. As in section 4.1 we employ the real Schur decomposition in order to change variables from the elements of G to the eigenvalues of G and some auxiliary variables. It is useful to use slightly different notation for the decomposition: $G = QRQ^T$ where $Q \in \mathbb{R}^{N\times N}$ is an orthogonal matrix, whose first row is chosen to be non-negative and the matrix $R \in \mathbb{R}^{N\times N}$ is block triangular of the form:

$$R = \begin{pmatrix} \lambda_1 & \cdots & r_{1k} & r_{1,k+1} & \cdots & r_{1,N} \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \lambda_k & r_{k,k+1} & \cdots & r_{k,N} \\ 0 & \cdots & 0 & Z_1 & \cdots & r_{k+1,N} \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & Z_l \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & Z_1 & & \\ & & & & \ddots & \\ 0 & & & & & Z_l \end{pmatrix}$$

Hence Λ is again block diagonal, $\lambda_1,\dots,\lambda_k$ are the real eigenvalues of G and for $j = 1,\dots,l$:

$$Z_j = \begin{pmatrix} x_j & b_j \\ -c_j & x_j \end{pmatrix}, \quad b_jc_j > 0, \quad b_j \le c_j \quad \text{and} \quad y_j = \sqrt{b_jc_j}.$$

Let S again denote the strictly upper triangular part of R. S is obtained from R by replacing the entries λ_j with zeros for $j = 1,\dots,k$ and replacing the matrices Z_m with zero matrices for $m = 1,\dots,l$, thus $R = \Lambda + S$. The Jacobian of the change of variables is given in theorem 1.3.17. As a result integrating out the matrix Q yields:

$$P_{N,k,l}^{\text{IndSpherical}}(\Lambda) = c_{N,k,l}^{\text{IndSpherical}} \frac{2^l \pi^{\frac{1}{4}N(N+1)}}{\prod_{j=1}^{N} \Gamma\left(\frac{j}{2}\right)} \left|\Delta\left(\{\lambda_j\}_{j=1,\ldots,k} \cup \{z_m, \bar{z}_m\}_{m=1,\ldots,l}\right)\right| \times$$
$$\prod_{j=1}^{k} \lambda_j^L \prod_{m=1}^{l} z^{2L}(b_m - c_m) \int_{(S)} \det\left(I_N + R^T R\right)^{-\frac{n+M}{2}} (dS). \quad (4.2.6)$$

The following lemma is needed:

Lemma 4.2.4.

$$\int_{(S)} \det\left(I_N + R^T R\right)^{-\frac{n+M}{2}} (dS) = \prod_{j=1}^{k} (1+\lambda_j^2)^{-\frac{n+L+1}{2}} \pi^{\frac{1}{2}(k-s)} \frac{\Gamma\left(\frac{n+L+1}{2}\right)}{\Gamma\left(\frac{n+L+k-j+1}{2}\right)} \times$$
$$\prod_{j=1}^{l} \det\left(I_2 + Z_m Z_m^T\right)^{-\frac{n+L+2}{2}} \pi^{N-2m-2} \frac{\Gamma\left(\frac{n+L+1}{2}\right)\Gamma\left(\frac{n+L+2}{2}\right)}{\Gamma\left(\frac{n+M-2s-1}{2}\right)\Gamma\left(\frac{n+M-2s}{2}\right)} \quad (4.2.7)$$

Proof. [FM11] We start with the entries of S that correspond to complex eigenvalue columns. In the following $S = S_N$ and the subscript shall denote the number of rows and columns. The integration can be performed by introducing a recurrence relation for the following integral:

$$I_{n,M,N} := \int_{(S_N)} \det\left(I_N + R^T R\right)^{-\frac{n+M}{2}} (dS_N). \quad (4.2.8)$$

For this purpose isolate the last two rows and columns of S_N:

$$\begin{pmatrix} R_{N-2} & \vec{u} \\ \vec{0}^T & Z_l \end{pmatrix}, \quad (4.2.9)$$

where $\vec{u}$ is of size $(N-2) \times 2$ and $\vec{0}$ is of size $2 \times (N-2)$. Then note:

$$I_N + R_N R_N^T = \begin{pmatrix} I_{N-2} + R_{N-2} R_{N-2} + \vec{u}\vec{u}^T & \vec{u} Z_l^T \\ Z_l \vec{u}^T & I_2 + Z_l Z_l^T \end{pmatrix}. \quad (4.2.10)$$

Using the block determinant formula:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det\left(A - BD^{-1}C\right) \quad (4.2.11)$$

yields:

$$
\begin{aligned}
&\det\left(I_N + R_N R_N^T\right) \\
=&\det(I_2 + Z_l Z_l^T)\det\left(I_{N-2} + R_{N-2}R_{N-2}^T + \vec{u}\vec{u}^T - \vec{u}Z_l^T(I_2 + Z_l Z_l^T)^{-1}Z_l\vec{u}^T\right) \\
=&\det(I_2 + Z_l Z_l^T)\det\left(I_{N-2} + R_{N-2}R_{N-2}^T\right)\times \\
&\det\left(I_{N-2} + (I_{N-2} + R_{N-2}R_{N-2}^T)^{-1}\vec{u}\big(I_2 - Z_l^T(I_2 + Z_l Z_l^T)^{-1}Z_l\big)\vec{u}^T\right).
\end{aligned} \tag{4.2.12}
$$

Furthermore applying the Woodbury matrix identity:

$$
\big(A + UCV\big)^{-1} = A^{-1} - A^{-1}U\big(C^{-1} + VA^{-1}U\big)^{-1}VA^{-1} \tag{4.2.13}
$$

with $A = C = I_2$, $U = Z_l^T$, $V = Z_l$ gives:

$$
\begin{aligned}
&\det\left(I_N + R_N R_N^T\right) = \det(I_2 + Z_l Z_l^T)\det\left(I_{N-2} + R_{N-2}R_{N-2}^T\right)\times \\
&\det\left(I_{N-2} + (I_{N-2} + R_{N-2}R_{N-2}^T)^{-1}\vec{u}(I_2 + Z_l Z_l^T)^{-1}\vec{u}^T\right) \\
=&\det(I_2 + Z_l Z_l^T)\det\left(I_{N-2} + R_{N-2}R_{N-2}^T\right)\times \\
&\det\left(I_2 + \vec{u}^T(I_{N-2} + R_{N-2}R_{N-2}^T)^{-1}\vec{u}(I_2 + Z_l Z_l^T)^{-1}\right).
\end{aligned} \tag{4.2.14}
$$

As a result:

$$
\begin{aligned}
I_{k,M,n} = \det(I_2 + Z_l Z_l^T)^{-\frac{n+M}{2}} \int_{(S_{N-2})} \det\left(I_{N-2} + R_{N-2}R_{N-2}^\dagger\right)^{-\frac{n+M}{2}} \times \quad (4.2.15) \\
\det\left(I_2 + (I_2 + Z_l Z_l^T)^{-\frac{1}{2}}\vec{u}^T(I_{N-2} + R_{N-2}R_{N-2}^T)^{-1}\vec{u}(I_2 + Z_l Z_l^T)^{-\frac{1}{2}}\right)^{-\frac{n+M}{2}}(dS_{N-2})
\end{aligned}
$$

A change of variables:

$$
\vec{v}_{N-2} = \left(I_{N-2} + R_{N-2}R_{N-2}^T\right)^{-\frac{1}{2}}\vec{u}_{N-2}(I_2 + Z_l Z_l^T)^{-\frac{1}{2}} \tag{4.2.16}
$$

with Jacobian:

$$
(d\vec{u}_{N-2}) = \det(I_2 + Z_l Z_l^T)^{\frac{N}{2}-1}\det\left(I_{N-2} + R_{N-2}R_{N-2}^T\right)^{-1}(d\vec{v}_{N-2}), \tag{4.2.17}
$$

then leads to:

$$
\begin{aligned}
I_{n,M,N} =& \det(I_2 + Z_l Z_l^T)^{-\frac{n+L+1}{2}} \times \\
&\int_{(\vec{v}_{N-2})} \left(I_2 + (\vec{v}_{N-2})^T(\vec{v}_{N-2})\right)^{-\frac{n+M}{2}}(d\vec{v}_{N-2})I_{n,M-2,N-2}.
\end{aligned} \tag{4.2.18}
$$

In addition set:

$$Q_{n,M,N-2} := \int_{(\vec{v}_{N-2})} \left(I_2 + \vec{v}_{N-2}^T \vec{v}_{N-2}\right)^{-\frac{n+M}{2}} (d\vec{v}_{N-2}). \tag{4.2.19}$$

Repeating this procedure once yields:

$$I_{n,M-2,N-2} = \det(I_2 + Z_{l-1} Z_{l-1}^T)^{-\frac{n+L+1}{2}} Q_{n,M-2,N-4} I_{n,M-4,N-4}. \tag{4.2.20}$$

Iterating this procedure l times thus gives:

$$I_{n,M,N} = \det\left(I_k + R_k R_k^T\right)^{-\frac{n+L+k}{2}} \prod_{m=1}^{l} \frac{Q_{n,M-2m,N-2m-2}}{\det(I_2 + Z_m Z_m^T)^{\frac{n+L+1}{2}}}. \tag{4.2.21}$$

In order to evaluate $Q_{n,M-2m,N-2m-2}$ we need to perform another change of variables $\vec{v}_{N-2m-2}^T \vec{v}_{N-2m-2} = W_{N-2m-2}$ with Jacobian:

$$(dv_{N-2m-2}) = \frac{\pi^{N-2m-2} \det(W_{N-2m-2})^{\frac{N-2m-5}{2}} (dW_{N-2m-2})}{\int_{(W_{N-2m-2})} \det(W_{N-2m-2})^{\frac{N-2m-5}{2}} \, \mathrm{e}^{-\operatorname{tr}(W_{N-2m-2})} (dW_{N-2m-2})}.$$

As a consequence:

$$Q_{n,M-2m,N-2m-2} = \pi^{N-2m-2} \frac{\int_{(W)} \det(W)^{\frac{N-2m-5}{2}} \det\left(I_2 + W\right)^{\frac{-n-M+2m}{2}} (dW)}{\int_{(W)} \det(W)^{\frac{N-2m-5}{2}} \, \mathrm{e}^{-\operatorname{tr}(W)} (dW)}.$$

Furthermore now change variables to the eigenvalues $x_1 \leq x_2$ of W. From [Mui82], Chapter 3 Eq. (22), we know:

$$(dW) = |x_1 - x_2| dx_1 dx_2 (dH). \tag{4.2.22}$$

Then:

$$Q_{n,M-2m,N-2m-2} = \pi^{N-2m-2} \frac{\int_0^\infty \int_0^\infty |x_1 - x_2| \frac{(x_1 x_2)^{\frac{N-2m-5}{2}}}{((1+x_1)(1+x_2))^{\frac{n+M-2m}{2}}} dx_1 dx_2}{\int_0^\infty \int_0^\infty |x_1 - x_2| (x_1 x_2)^{\frac{N-2m-5}{2}} \, \mathrm{e}^{-x_1 - x_2} \, dx_1 dx_2}.$$

Again we need to change variables in the numerator as follows $y_i = \frac{x_i}{1+x_i}$ with $\frac{1}{1+x_i} = 1 - y_i$ and Jacobian $\frac{1}{(1-y_i)^2}$. Then the two integrals reduce to known variants of the Selberg integral and:

$$Q_{n,M-2m,N-2m-2} = \pi^{N-2m-2} \frac{\Gamma\left(\frac{n+L+1}{2}\right) \Gamma\left(\frac{n+L+2}{2}\right)}{\Gamma\left(\frac{n+M-2m-1}{2}\right) \Gamma\left(\frac{n+M-2m}{2}\right)}. \tag{4.2.23}$$

The second part of the calculation involves integrating out the columns corresponding to real eigenvalue columns. As before we isolate the last row and column of the matrix:

$$R_k = \left(\begin{array}{cc} R_{k-} & \vec{u}_{k-1} \\ 0^T & \lambda_k \end{array} \right), \tag{4.2.24}$$

where u_{k-1} is of size $(k-1) \times 1$ and 0^T is of size $1 \times (k-1)$. As before we can write:

$$\begin{aligned} &\det\left(I_k + R_k R_k^T\right) \\ =&(1+\lambda_k^2)\det\left(I_{k-1} + R_k R_k^T\right)\Big(1 + (1+\lambda_k^2)^{-1}\vec{u}_{k-1}^T(I_{k-1} + R_k R_k^T)^{-1}\vec{u}_{k-1}\Big). \end{aligned} \tag{4.2.25}$$

Consequently:

$$\begin{aligned} &\int_{(\vec{u}_{k-1})} \det\left(I_k + R_k R_k^T\right)^{-\frac{n+L+k}{2}} (d\vec{u}_{k-1}) = \frac{(1+\lambda_k^2)^{-\frac{n+L+k}{2}}}{\det\left(I_{k-1} + R_{k-1}R_{k-1}^T\right)^{\frac{n+L+k}{2}}} \times \\ &\int_{(\vec{u}_{k-1})} \left(1 + (1+\lambda_k^2)^{-1}\vec{u}_{k-1}^T(I_{k-1} + R_k R_k^T)^{-1}\vec{u}_{k-1}\right)^{-\frac{n+L+k}{2}} .(d\vec{u}_{k-1}) \end{aligned} \tag{4.2.26}$$

Now change variables:

$$\vec{v}_{k-1} = \left(I_{k-1} + R_{k-1}R_{k-1}^T\right)^{-\frac{1}{2}} \vec{u}_{k-1}(1+\lambda_{k-1})^{-\frac{1}{2}} \tag{4.2.27}$$

with Jacobian:

$$(d\vec{v}_{k-1}) = \det\left(I_{k-1} + R_{k-1}R_{k-1}^T\right)^{-\frac{1}{2}}(1+\lambda_{k-1})^{-\frac{1}{2}(k-1)}. \tag{4.2.28}$$

Hence:

$$\begin{aligned} &\int_{(\vec{u}_{k-1})} \det\left(I_k + R_k R_k^T\right)^{-\frac{n+L+k}{2}} (d\vec{u}_{k-1}) \\ =&\frac{(1+\lambda_k^2)^{-\frac{n+L+1}{2}}}{\det\left(I_{k-1} + R_{k-1}R_{k-1}^T\right)^{\frac{n+L+k-1}{2}}} \int_{(\vec{v}_{k-1})} \left(1 + \vec{v}_{k-1}^T\vec{v}_{k-1}\right)^{\frac{n+L+k}{2}} (d\vec{v}_{k-1}). \end{aligned} \tag{4.2.29}$$

Iterating the procedure then yields:

$$\begin{aligned} &\int_{(S_k)} \frac{1}{\det\left(I_k + R_k R_k^T\right)^{\frac{n+L+k}{2}}} (dS_k) = \prod_{j=1}^{k} \frac{1}{(1+\lambda_j^2)^{\frac{n+L+1}{2}}} \\ &\prod_{j=1}^{k-1} \int_{(\vec{v}_{k-j})} \frac{1}{\left(1 + \vec{v}_{k-j}^T\vec{v}_{k-j}\right)^{\frac{n+L+k-j+1}{2}}} (d\vec{v}_{k-j}). \end{aligned} \tag{4.2.30}$$

Finally we need to evaluate:

$$\tilde{Q}_{n,M,k-j} = \int_{(\vec{v}_{k-j})} \left(1 + \vec{v}_{k-j}^T \vec{v}_{k-j}\right)^{-\frac{n+L+k-j+1}{2}} .(d\vec{v}_{k-j}) \tag{4.2.31}$$

As before we change variables in order to evaluate $\tilde{C}_{n,M,k-j}$ we need to perform another change of variables $\vec{v}_{k-j}^T \vec{v}_{k-j} = w$ with Jacobian:

$$(dv_{k-j}) = \frac{\pi^{\frac{1}{2}(k-j)}}{\Gamma\left(\frac{k-j}{2}\right)} w^{\frac{k-j-2}{2}} dw. \tag{4.2.32}$$

As a consequence:

$$\tilde{Q}_{n,M,k-j} = \frac{\pi^{\frac{1}{2}(k-j)}}{\Gamma\left(\frac{k-j}{2}\right)} \int_0^\infty \frac{w^{\frac{k-j-2}{2}}}{(1+w)^{\frac{n+L+k-j+1}{2}}} dw = \pi^{\frac{1}{2}(k-j)} \frac{\Gamma\left(\frac{n+L+1}{2}\right)}{\Gamma\left(\frac{n+L+k-j+1}{2}\right)}, \tag{4.2.33}$$

which proves the lemma. □

As a result:

$$p_{N,k,l}^{\text{IndSpherical}}(\Lambda) = c_{N,k,l}^{\text{IndSpherical}} \frac{2^l \pi^{\frac{1}{4}N(N+1)}}{\prod_{j=1}^N \Gamma\left(\frac{j}{2}\right)} \prod_{j=1}^k \pi^{\frac{1}{2}(k-j)} \frac{\Gamma\left(\frac{n+L+1}{2}\right)}{\Gamma\left(\frac{n+L+k-j+1}{2}\right)} \times$$
$$\prod_{m=0}^{l-1} \pi^{N-2m-2} \frac{\Gamma\left(\frac{n+L+1}{2}\right)\Gamma\left(\frac{n+L+2}{2}\right)}{\Gamma\left(\frac{n+M-2m-1}{2}\right)\Gamma\left(\frac{n+M-2m}{2}\right)} \left|\Delta\left(\{\lambda_j\}_{j=1}^k \cup \{z_m, \bar{z}_m\}_{m=1}^l\right)\right| \times$$
$$\prod_{j=1}^k \lambda_j^L (1+\lambda_j^2)^{-\frac{n+L+1}{2}} \prod_{m=1}^l z_m^{2L} \det\left(I_2 + Z_{m+1} Z_{m+1}^T\right)^{-\frac{n+L+2}{2}} (b_m - c_m). \tag{4.2.34}$$

Finally using the change of variables from equation (4.1.8) as well as noting:

$$\begin{aligned}
&\int \det\left(I_2 + Z_m Z_m^T\right)^{-\frac{n+L+2}{2}} \\
&= 2^3 \int_0^{1+x_m^2+y_m^2} \frac{\delta_m y_m}{\sqrt{\delta_m^2 + 4y_m^2}} \left[(1+x_m^2+y_m^2)^2 - \delta_m^2\right]^{-\frac{n+L+2}{2}} d\delta_m \\
&= 2^3 \int_{2|y_m|}^{\sqrt{(1+x_m^2+y_m^2)^2+4y_m^2}} y_m \left[(1+x_m^2+y_m^2)^2 - 4y_m^2 + t_m^2\right]^{-\frac{n+L+2}{2}} dt_m \\
&= 2^3 \operatorname{Im}(z_m) |1+z_m^2|^{-n-L-1} \int_{\frac{2|\operatorname{Im}(z_m)|}{|1+z_m^2|}}^1 (1+u^2)^{-\frac{n+L+2}{2}} du
\end{aligned} \tag{4.2.35}$$

then proves the statement. □

Setting the parameters $n = M = N$ one recovers the eigenvalue jpdf of the real spherical ensemble, which was calculated in [FM11]. Note that the weight function in the eigenvalue jpdf of the real induced spherical ensemble differs from the weight function of the real spherical ensemble by the factor λ^L , which is introduced by the inducing procedure. Again the probability of finding eigenvalues close to the origin is small, as eigenvalues are repulsed from the origin. The strength of repulsion is controlled by the rectangularity parameter L.

4.2.2 The real induced Jacobi ensemble: Eigenvalue jpdf

Applying the inducing procedure to a real rectangular truncation $A \in \mathbb{R}^{M\times N}$ as defined in theorem 2.0.29 yields a random matrix G pertaining to the so-called real induced Jacobi ensemble.

Definition 4.2.5. *For $K \geq M+N$ the real induced Jacobi ensemble with parameters K, M is specified by the following probability measure on the space of $N\times N$ matrices: $\mathrm{d}\,\mu^{Induced}_{Jacobi,1}(G) = P^{Induced}_{Jacobi,1}(G)(dG)$, with*

$$P^{Induced}_{Jacobi,1}(G) = \gamma^{IndJacobi,1}_{K,M,N} \det\left(GG^T\right)^{\frac{L}{2}} \det\left(I_N - GG^T\right)^{\frac{K-M-N-1}{2}}, \qquad (4.2.36)$$

where $L := M - N \geq 0$.

In the case $K \leq N+M$ the element jpdf of a real induced Jacobi matrix contains again singular terms. Setting the parameter $L=0$ one recovers the matrix measure of truncations of random orthogonal matrices [KSŻ10] Note that the parameters:

$$l_M := K - M \qquad l_N := K - N\,, \qquad (4.2.37)$$

denote the number of rows (l_M) and columns (l_N), that are deleted from the initial orthogonal matrix used to generate the ensemble. The name of the ensemble refers to the fact, that the induced measure boasts a Jacobi weight. For $K < N+M$ the matrix measure of the induced Jacobi ensemble contains δ functions and thus is singular.

Lemma 4.2.6. *The induced Jacobi ensemble is correctly normalized for $K > N+M$ using:*

$$\gamma^{IndJacobi,1}_{K,M,N} = \pi^{-N^2} \prod_{j=1}^{N} \frac{\Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{l_N+j+1}{2}\right)}{\Gamma\left(\frac{L+j}{2}\right)\Gamma\left(\frac{l_M-N+j+1}{2}\right)}. \qquad (4.2.38)$$

Thus we can proceed with deriving the eigenvalue distribution of an induced real Jacobi matrix. In the following it is as before assumed, that A has k real ordered eigenvalues: $\lambda_1 \geq \ldots \geq \lambda_k$, while $l = \frac{N-k}{2}$ denotes the number of complex conjugate eigenvalue pairs $x_1 \pm iy_1, \ldots, x_l \pm iy_l$ ordered by their real part. We adopt the convention that $y_j > 0$ for all j. In the case of two complex eigenvalues with identical real part the eigenvalue pairs are ordered by the imaginary part.

As for the case of the complex induced Jacobi measure (and for its counterpart the square truncations of orthogonal matrices) the matrix measure of the real induced Jacobi ensemble only exists if $K \geq N + M$, meaning that, a sufficient number of rows and columns need to be deleted from the orthogonal matrix used to generate the complex induced Jacobi matrix. Nevertheless, even though the matrix measure is singular for $K < M + N$, it is still possible to derive the distribution of eigenvalues for all possible values of K, M, N. In order to avoid the singularity of the matrix measure in the derivation of the eigenvalue jpdf, we start with the joint distribution of the matrices A, C from theorem 2.0.28. Then using the quadratization from chapter 2 a change of variable is applied such that, we arrive at the joint distribution of G, W, C. Here G denotes the square quadratization of A. Incidently G is a real induced Jacobi matrix and by using the real Schur decomposition and integrating out W as well as C it is possible to derive the eigenvalue jpdf of an induced Jacobi matrix for all possible integer values of K, M, N.

Theorem 4.2.7. *The eigenvalue jpdf of a real induced Jacobi matrix with k real eigenvalues and l pairs of complex conjugated eigenvalues is given by:*

$$P_{N,k,l}^{IndJacobi}(\lambda_1, \ldots, \lambda_k, z_1, \ldots, z_l) = c_{N,k,l}^{IndJacobi} \left| \Delta\left(\{\lambda_j\}_{j=1}^k \cup \{z_m, \bar{z}_m\}_{m=1}^l\right) \right| \times \\ \prod_{j=1}^k w_{IndJacobi,1}(\lambda_j) \prod_{m=1}^l \operatorname{Im}(z_m) w_{IndJacobi,1}(z_m) w_{IndJacobi,1}(\bar{z}_m), \tag{4.2.39}$$

where

$$w_{IndJacobi,1}(z) = z^L |1 - z^2|^{\frac{l_M - 2}{2}} \Big(\int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^1 (1 - u^2)^{\frac{l_M - 3}{2}} du \Big)^{\frac{1}{2}} \tag{4.2.40}$$

$$c_{N,k,l}^{IndJacobi} = \left[\frac{2}{\pi} \frac{(2\pi)^{l_M}}{(l_M - 2)!} \right]^{\frac{N}{2}} 2^{-k} \pi^{-\frac{1}{2} N l_M} \prod_{j=1}^N \frac{\Gamma(\frac{l_N + j}{2})}{\Gamma(\frac{l_M + j}{2})} \tag{4.2.41}$$

as well as $\lambda_j \in [0, 1]$ for $j = 1, \ldots, k$ and $z_m \in \mathbb{D}_+ := \{z \,\big|\, |z| \leq 1 \wedge 0 \leq \arg(z) \leq \pi\}$ for $m = 1, \ldots, l$. Integrating the partial eigenvalue jpdf $P_{N,k,l}^{IndJacobi}$ over $[-1, 1]^k \times \mathbb{D}_+^{2l}$ gives $p_{N,k}^{IndJacobi}$, the probability that G has k real eigenvalues.

Proof. As the element jpdf for $K < M + N$ is singular we start from

$$P(A, C) = c_{\text{Stief},1}\, \delta\big(A^T A + C^T C - I_N\big) \tag{4.2.42}$$

We apply the quadratization procedure by changing variables form the rectangular matrix $A = \begin{pmatrix} Y \\ Z \end{pmatrix}$ to W, G where $W^T A = \begin{pmatrix} G \\ O \end{pmatrix}$ and thus $A = W \begin{pmatrix} G \\ 0 \end{pmatrix}$.The matrix W is orthogonal and the decomposition is unique if W is chosen from the coset $O(M)/\big(O(N) \times O(M-N)\big)$. The Jacobian of this change of variables is then given by 2:

$$|J| = \det\big(GG^T\big)^{\frac{L}{2}}. \tag{4.2.43}$$

Furthermore note:

$$A^T A = \begin{pmatrix} G & O \end{pmatrix} W^T W \begin{pmatrix} G \\ O \end{pmatrix} = G^T G. \tag{4.2.44}$$

As a result:

$$P(W, G, C) = c_{\text{Stief}} \det\big(GG^T\big)^{\frac{L}{2}} \delta\big(G^T G + C^T C - I_N\big). \tag{4.2.45}$$

Integrating out the matrix W, then yields:

$$P(G, C) = c_{\text{Stief}} \frac{\text{Vol}(O(M))}{\text{Vol}(O(L))\,\text{Vol}(O(N))} \det\big(GG^T\big)^{\frac{L}{2}} \delta\big(G^T G + C^T C - I_N\big). \tag{4.2.46}$$

Now we change variables, again using the real Schur decomposition $G = QRQ^T$ from 1.3.15, where $Q \in \mathbb{R}^{N\times N}$ is an orthogonal matrix, whose first row is chosen to be non-negative and the matrix $R \in \mathbb{R}^{N\times N}$ is block triangular of the form: $R = \Lambda + S$. Again it is convinient to use slightly different notation. Let Λ be block diagonal of the form:

$$\begin{pmatrix} \Lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \Lambda_{\frac{k}{2}} & & & \\ & & & \Lambda_{\frac{k}{2}+1} & & \\ & & & & \ddots & \\ 0 & & & & & \Lambda_{\frac{N}{2}} \end{pmatrix} \tag{4.2.47}$$

containing for $j=1,\dots,\frac{k}{2}$ as well as for $m=\frac{k}{2}+1,\dots,\frac{N}{2}$ the 2×2 blocks:

$$\Lambda_j = \begin{pmatrix} \lambda_j & r_j \\ 0 & \lambda_{j+1} \end{pmatrix},\ \Lambda_m = \begin{pmatrix} x_m & b_m \\ -c_m & x_m \end{pmatrix},\ b_jc_j>0,\ b_j\le c_j \text{ and } \quad y_j=\sqrt{b_jc_j}$$

on its block diagonal, where $\lambda_1,\dots,\lambda_k$ are real eigenvalues of G and $z_m = x_m + iy_m$, $\bar{z}_m = x_m - iy_m$ are the complex conjugated eigenvalue pairs of G. The matrix S is block upper triangular and for the purpose of the following calculations is also divided into 2×2 blocks S_{ij} with $i=1,\dots,N$, $j<i$:

$$S = \begin{pmatrix} 0 & S_{12} & & \cdots & S_{1\frac{N}{2}} \\ 0 & 0 & S_{23} & \cdots & S_{1\frac{N}{2}} \\ \vdots & & \ddots & \ddots & S_{\frac{N}{2}-1,\frac{N}{2}} \\ 0 & \cdots & & 0 & 0 \end{pmatrix}.$$

As a consequence:

$$P(\Lambda,Q,S,C) = c_{\text{Stief},1}\frac{\operatorname{Vol}(O(M))}{\operatorname{Vol}(O(L))\operatorname{Vol}(O(N))}\det\left(GG^T\right)^{\frac{L}{2}} \tag{4.2.48}$$
$$\left|\Delta\left(\{\lambda_j\}_{j=1}^k \cup \{z_m,\bar{z}_m\}_{m=1}^l\right)\right| 2^l \prod_{j=1}^{l}(b_j-c_j)\delta\left(\left(\Lambda^T+S^T\right)\left(\Lambda+S\right)+C^TC-I_N\right).$$

Integrating out the orthogonal matrix Q then gives:

$$P(\Lambda,S,C) = c_{\text{Stief}}\frac{\operatorname{Vol}(O(M))}{2^N\operatorname{Vol}(O(L))}\det\left(GG^T\right)^{\frac{L}{2}} \tag{4.2.49}$$
$$\left|\Delta\left(\{\lambda_j\}_{j=1}^k \cup \{z_m,\bar{z}_m\}_{m=1}^l\right)\right| 2^l \prod_{j=1}^{l}(b_j-c_j)\delta\left(\left(\Lambda^T+S^T\right)\left(\Lambda+S\right)+C^TC-I_N\right).$$

We need the following lemma:

Lemma 4.2.8.

$$\int_{(C)}\int_{(S)}\delta\left(\left(\Lambda^T+S^T\right)\left(\Lambda+S\right)+C^TC-I_N\right)(dS)(dC) = \prod_{j=1}^{\frac{N}{2}}\det\left(I_2-\Lambda_j^T\Lambda_j\right)^{\frac{L-3}{2}}$$

Proof. It is helpful to divide the matrix C into $\frac{N}{2}$ subblocks of size $l_M\times 2$, $C=\left(C_1,\dots,C_{\frac{N}{2}}\right)$. The proof of this lemma is inspired from [KSŻ10, SŻ00], though it varies in details. It is structurally identical to the proof of lemma 3.2.7. The idea is to first integrate out the upper block-triangular matrix S, by integrating each of its blocks, starting from the leftmost block in the first row and then moving row by row. Now the delta function with block matrix multiplication give the conditions for $i = 1, \ldots, \frac{N}{2}$ and $j = i+1, \ldots, \frac{N}{2}$:

$$\Lambda_i^T S_{ij} + C_i^T C_j + \sum_{n<i} S_{ni}^T S_{nj} = 0 \tag{4.2.50}$$

$$\Lambda_i^T \Lambda_i + C_i^T C_i + \sum_{n<i} S_{ni}^T S_{ni} - I_2 = 0. \tag{4.2.51}$$

Especially the first row gives for $j = 2, \ldots, \frac{N}{2}$:

$$\Lambda_1^T S_{1j} + C_1^T C_j = 0. \tag{4.2.52}$$

The first step is changing variables for $j = 2, \ldots, \frac{N}{2}$:

$$S_{1j}^{(1)} = \Lambda_1^{-T} S_{1j} \tag{4.2.53}$$

with Jacobian:

$$|J_1| = \prod_{j=2}^{\frac{N}{2}} \det(\Lambda_1)^{-2}. \tag{4.2.54}$$

Note that $S_{1j}^{(1)} = -C_1^T C_j$. Using this relation for $i = 2, \ldots, \frac{N}{2}$ yields:

$$\begin{aligned} &\Lambda_i^T \Lambda_i + C_i^T C_1 \Lambda_1^{-1} \Lambda_1^{-T} C_1^T C_i + C_i^T C_i + \sum_{1<n<i} S_{ni}^T S_{ni} - I_2 = 0 \\ \Leftrightarrow\ &\Lambda_i^T \Lambda_i + C_i^T \big(C_1 \Lambda_1^{-1} \Lambda_1^{-T} C_1^T + I_{l_M}\big) C_i + \sum_{1<n<i} S_{ni}^T S_{ni} - I_2 = 0 \end{aligned} \tag{4.2.55}$$

as well as:

$$\begin{aligned} &\Lambda_i^T S_{ij} + C_j^T C_1 \Lambda_1^{-1} \Lambda_1^{-T} C_1^T C_i + C_j^T C_i + \sum_{1<n<i} S_{nj}^T S_{ni} = 0 \\ \Leftrightarrow\ &\Lambda_i^T S_{ij} + C_j^T \big(C_1 \Lambda_1^{-1} \Lambda_1^{-T} C_1^T + I_{l_M}\big) C_i + \sum_{1<n<i} S_{nj}^T S_{ni} = 0 \end{aligned} \tag{4.2.56}$$

for $j > i$. We change variables again for $i = 2, \ldots, \frac{N}{2}$:

$$C_i^{(1)} = \sqrt{X_1} C_i \tag{4.2.57}$$

with $X_1 = C_1 \Lambda_1^{-1} \Lambda_1^{-T} C_1^T + I_{l_M}$ and Jacobian:

$$|\hat{J}_1| = \prod_{j=1}^{\frac{N}{2}} \det(X_1)^{-1}. \tag{4.2.58}$$

Furthermore:

$$\det(X_1) = \det\big(C_1(\Lambda_1^T\Lambda_1)^{-1}C_1^T + I_{l_M}\big). \tag{4.2.59}$$

Applying Sylvester's determinant theorem then gives:

$$\det(X_1) = \det\big((\Lambda_1^T\Lambda_1)^{-1}C_1^T C_1 + I_2\big). \tag{4.2.60}$$

Furthermore from equation (4.2.51):

$$\det(X_1) = \det\big((\Lambda_1^T\Lambda_1)^{-1}(I_2 - \Lambda_1^T\Lambda_1) + I_2\big) = \det(\Lambda_1)^{-2}. \tag{4.2.61}$$

Thus the Jacobian:

$$|\hat{J}_1| = \prod_{j=2}^{\frac{N}{2}} \det(\Lambda_1)^2 \tag{4.2.62}$$

cancels the Jacobian $|J_1|$ of the previous change of variables. As a result for $i = 2, \ldots, \frac{N}{2}$, $j > i$:

$$\Lambda_i^T S_{ij} + (C_i^{(1)})^T C_j^{(1)} + \sum_{1<n<i} S_{ni}^T S_{nj} = 0 \tag{4.2.63}$$

$$\Lambda_i^T \Lambda_i + (C_i^{(1)})^T C_i^{(1)} + \sum_{1<n<i} S_{ni}^T S_{ni} - I_2 = 0. \tag{4.2.64}$$

Now the second row gives for $j = 3, \ldots, \frac{N}{2}$:

$$\Lambda_2^T S_{2j} + (C_2^{(1)})^T C_j^{(1)} = 0. \tag{4.2.65}$$

Again we change variables for $j = 3, \ldots, \frac{N}{2}$:

$$S_{2j}^{(1)} = \Lambda_2^{-T} S_{2j} \tag{4.2.66}$$

with Jacobian:

$$|J_2| = \prod_{j=3}^{\frac{N}{2}} \det(\Lambda_2)^{-2}. \tag{4.2.67}$$

Note that $S_{2j}^{(1)} = -(C_2^{(1)})^T C_j^{(1)}$. Using this relation for $i = 3, \ldots, \frac{N}{2}$ yields:

$$\begin{aligned} &\Lambda_i^T\Lambda_i + (C_i^{(1)})^T C_2^{(1)} \Lambda_2^{-1}\Lambda_2^{-T}(C_2^{(1)})^T C_i^{(1)} + (C_i^{(1)})^T C_i^{(1)} + \sum_{2<k<i} S_{ki}^T S_{ki} - I_2 = 0 \\ \Leftrightarrow\ &\Lambda_i^T\Lambda_i + (C_i^{(1)})^T \big(C_2^{(1)}\Lambda_2^{-1}\Lambda_2^{-T}(C_2^{(1)})^T + I_{l_M}\big) C_i^{(1)} + \sum_{2<k<i} S_{ki}^T S_{ki} - I_2 = 0 \end{aligned} \tag{4.2.68}$$

as well as:

$$\Lambda_i^T S_{ij} + (C_j^{(1)})^T C_2^{(1)} \Lambda_2^{-1} \Lambda_2^{-T} (C_2^{(1)})^T C_i^{(1)} + (C_j^{(1)})^T C_i^{(1)} + \sum_{2<k<i} S_{kj}^T S_{ki} = 0$$
$$\Leftrightarrow \Lambda_i^T S_{ij} + (C_j^{(1)})^T \big(C_2^{(1)} \Lambda_2^{-1} \Lambda_2^{-T} (C_2^{(1)})^T + I_{l_M} \big) C_i^{(1)} + \sum_{2<k<i} S_{kj}^T S_{ki} = 0 \tag{4.2.69}$$

for $j > i$. We change variables again for $i = 3, \ldots, \frac{N}{2}$:

$$C_i^{(2)} = \sqrt{X_2} C_i^{(1)} \tag{4.2.70}$$

with $X_2 = C_2^{(1)} \Lambda_2^{-1} \Lambda_2^{-T} (C_2^{(1)})^T + I_{l_M}$ and Jacobian:

$$|\hat{J}_2| = \prod_{j=3}^{\frac{N}{2}} \det(X_2)^{-1}. \tag{4.2.71}$$

As before applying Sylvester's theorem as well as equation (4.2.64) give $|\hat{J}_2| = \prod_{j=3}^{\frac{N}{2}} \det(\Lambda_2)^2$. Again the Jacobian of the previous change of variables cancels the Jacobian $|\hat{J}_2|$. Consequently for $i = 3, \ldots, \frac{N}{2}$, $j > i$:

$$\Lambda_i^T S_{ij} + (C_i^{(2)})^T C_j^{(2)} + \sum_{2<k<i} S_{ki}^T S_{kj} = 0 \tag{4.2.72}$$

$$\Lambda_i^T \Lambda_i + (C_i^{(2)})^T C_i^{(2)} + \sum_{2<k<i} S_{ki}^T S_{ki} - I_2 = 0. \tag{4.2.73}$$

Repeating this procedure for all rows then yields:

$$\int_{(C)} \int_{(S)} \delta\big((\Lambda^T + S^T)(\Lambda + S) + C^T C - I_N \big) (dS)(dC) = \prod_{j=1}^{\frac{N}{2}} \int_{(C_j)} \delta\big(C_j^T C_j + \Lambda_j^T \Lambda_j - I_2 \big) (dC_j). \tag{4.2.74}$$

The last integral can be solved by a final change of variables:

$$D_j = C_j \sqrt{I_2 - \Lambda_j^T \Lambda_j} \tag{4.2.75}$$

with Jacobian $\det\big(I_2 \Lambda_j^T \Lambda_j \big)^{\frac{l_M}{2}}$, whereas the delta function contributes a factor of $\det\big(I_2 - \Lambda_j^T \Lambda_j \big)^{-\frac{3}{2}}$ $\square$

Applying lemma 4.2.8 thus gives

$$p_{N,k,l}^{\mathrm{IndJacobi}}(\Lambda) = c_{\mathrm{Stief},1} \frac{\mathrm{Vol}(O(M))}{2^N \,\mathrm{Vol}(O(L))} \det\left(\Lambda\Lambda^T\right)^{\frac{L}{2}} \times$$

$$\left|\Delta\left(\{\lambda_j\}_{j=1}^k \cup \{z_m, \bar{z}_m\}_{j=1}^l\right)\right| 2^l \prod_{j=1}^{l}(b_j - c_j) \prod_{j=1}^{\frac{N}{2}} \det\left(I_2 - \Lambda_j^T \Lambda_j\right)^{\frac{l_M - 3}{2}} \tag{4.2.76}$$

Note that for $j \leq k$ the following holds:

$$\det\left(I_2 - \Lambda_j^T \Lambda_j\right) = (1 - \lambda_j^2)(1 - \lambda_{j+1}^2) - r_j^2 \geq 0. \tag{4.2.77}$$

In addition:

$$\int_{-\sqrt{(1-\lambda_j^2)(1-\lambda_{j+1}^2)}}^{\sqrt{(1-\lambda_j^2)(1-\lambda_{j+1}^2)}} \left[(1 - \lambda_j^2)(1 - \lambda_{j+1}^2) - r_j^2\right]^{\frac{L-3}{2}} \tag{4.2.78}$$

$$= 2\left[(1 - \lambda_j^2)(1 - \lambda_{j+1}^2)\right]^{\frac{L-2}{2}} \int_0^1 (1 - u^2)^{\frac{l_M - 3}{2}}. \tag{4.2.79}$$

Moreover for $j > k$:

$$\det\left(I_2 - \Lambda_j^T \Lambda_j\right) = (1 - x_j^2 - b_j c_j) - (b_j - c_j)^2. \tag{4.2.80}$$

Applying the change of variables from 4.1.8 then yields for $j > k$:

$$\det\left(I_2 - \Lambda_j^T \Lambda_j\right) = (1 - x_j^2 - y_j^2) - \delta_j^2, \tag{4.2.81}$$

as well as:

$$\begin{aligned}
\int \det\left(I_2 - \Lambda_j^T \Lambda_j\right)^{\frac{l_M-3}{2}} &= 2^3 \int_0^{1-x_j^2-y_j^2} \frac{\delta_j y_j}{\sqrt{\delta_j^2 + 4y_j^2}} \left[(1 - x_j^2 - y_j^2)^2 - \delta_j^2\right]^{\frac{l_M-3}{2}} d\delta_j \\
&= 2^3 \int_{2|y_j|}^{\sqrt{(1-x_j^2-y_j^2)^2+4y_j^2}} y_j \left[(1 - x_j^2 - y_j^2)^2 + 4y_j^2 - t_j^2\right]^{\frac{l_M-3}{2}} dt_j \\
&= 2^3 y_j |1 - z_j^2|^{l_M - 2} \int_{\frac{2|y_j|}{|1-z_j^2|}}^1 (1 - u^2)^{\frac{l_M-3}{2}} du.
\end{aligned}$$

□

Again setting the rectangularity parameter $L = 0$ we recover the eigenvalue jpdf of a square trunction of a random orthogonal matrix, [KSŻ10]. The inducing procedure results in the additional factor λ_j^L in the weight function of the ensemble. Again it should be noted that the eigenvalue jpdf is valid for $K \geq N + M$

as well as $K < N + M$. However the derivation of theorem 3.2.6 is only valid for integer values of L and $K - M - N$, as it relies on being able to apply the quadratization procedure from chapter 2 to a rectangular matrix of dimension $M \times N$.

4.2.3 The characteristic average

As seen in section 4.1.3 the characteristic average gives access to the relevant skew-orthogonal polynomials and thus to the entries of the Pfaffian kernel describing all (K', L')-correlation functions. In the case of the real induced spherical and Jacobi ensemble:

Theorem 4.2.9. *(a) For the real induced Jacobi ensemble of $m \times m$ matrices with parameters K, M the characteristic average is given by:*

$$\begin{aligned}&\langle \det \big((A - zI_m)(A - vI_m)\big)\rangle_{A_m}^{\mathit{IndJacobi}} \\ &= \sum_{j=0}^{m} \frac{\Gamma(K - N + j + 1)\Gamma(L + m + 1)}{\Gamma(L + j + 1)\Gamma(K - N + m + 1)} (zv)^j.\end{aligned} \tag{4.2.82}$$

(b) For the real induced spherical ensemble of $m \times m$ matrices with parameters n, M the characteristic average is given by:

$$\begin{aligned}&\langle \det \big((A - zI_m)(A - vI_m)\big)\rangle_{A_m}^{\mathit{IndSpherical}} \\ &= \sum_{j=0}^{m} \frac{\Gamma(n - m - 1)\Gamma(L + m + 1)}{\Gamma(L + j + 1)\Gamma(n - j - 1)} (zv)^j.\end{aligned} \tag{4.2.83}$$

Proof. As in the case of the real induced Ginibre ensemble we apply theorem 4.1.12. Furthermore notice that for both ensembles the average $\langle\rangle_{A_m}$ is invariant with respect to orthogonal transformation. This is due to the fact that the determinant as well as the probability measures are invariant with respect to orthogonal transformation. We may write then for an orthogonal matrix $Q \in O(N)$ and $I \in \{\text{IndJacobi}, \text{IndSpherical}\}$:

$$\langle \det \big((A - zI_m)(A - vI_m)\big)\rangle_{A_m}^{I} = \big\langle \langle \det \big((AQ - zI_m)(AQ - vI_m)\big)\rangle_{O(N)} \big\rangle_{A_m}^{I}. \tag{4.2.84}$$

Consequently we can apply theorem 4.1.12 and obtain:

$$\Big\langle \det\big((A - zI_m)(A - vI_m)\big)\Big\rangle^I_{A_m} = \Big\langle \sum_{j=0}^{m} \frac{\epsilon_j(AA^T)}{\binom{m}{k}}(zv)^{m-j}\Big\rangle^I_{A_m}$$
$$= \sum_{j=0}^{m} \frac{\langle \epsilon_j(AA^T)\rangle^I_{A_m}}{\binom{m}{j}}(zv)^{m-j}, \qquad (4.2.85)$$

where we used that $\epsilon_j(I_m) = \binom{m}{j}$. Thus it remains to calculate the average over the symmetric polynomials in the eigenvalues of AA^T. The following calculation is only valid for $K \geq M + N$. For proof of part (a) of this theorem in the special case $K < N + M$ see section D.3 in the appendix.

$$\langle \epsilon_j(AA^T)\rangle^{\text{IndJacobi}}_{A_m} \qquad (4.2.86)$$
$$= \gamma^{\text{IndJacobi}}_{K,M,m} \int_{(A)} \sum_{1 \leq i_1 < \cdots < i_j \leq m} l_{i_1} \cdots l_{i_j} \det(AA^T)^{\frac{L}{2}} \det\big(I_m - AA^T\big)^{\frac{l_M - m - 1}{2}} (dA)$$
$$\langle \epsilon_j(A^TA)\rangle^{\text{IndSpherical}}_{A_m} \qquad (4.2.87)$$
$$= C^{\text{IndSpherical}}_{M,m,n} \int_{(A)} \sum_{1 \leq i_1 < \cdots < i_j \leq m} l_{i_1} \cdots l_{i_j} \frac{\det(AA^T)^{\frac{L}{2}}}{\det\big(I_m + AA^T\big)^{\frac{n+L+m}{2}}} (dA).$$

We thus change variables to the singular value decomposition $A_m = U\Sigma V$ with singular values $0 \leq \sigma_1 \leq \ldots \leq \sigma_m$ of A_m and the Jacobian $\prod_{i<j} |\sigma_i^2 - \sigma_j^2|$. Both averages are symmetric in the singular values. Each term in the sum of singular values is of length j and all terms are distinct. Thus there are $\binom{m}{j}$ terms in the sum. Moreover we remove the ordering of the singular values which gives a factor of $m!$. As a result:

$$\langle \epsilon_j(AA^T)\rangle^{\text{IndJacobi}}_{A_m} = \gamma^{\text{IndJacobi}}_{K,M,m} \binom{m}{j} \frac{1}{m!} \operatorname{Vol}(O(m)) \operatorname{Vol}(O[m]) \times$$
$$\int_{(\Sigma)} \prod_{i_1 < i_2} |\sigma_{i_1}^2 - \sigma_{i_2}^2| \prod_{i=1}^{j} \sigma_i^2 \det(\Sigma)^L \det\big(I_m - \Sigma\Sigma^T\big)^{\frac{l_M - m - 1}{2}} (d\Sigma) \qquad (4.2.88)$$
$$= \gamma^{\text{IndJacobi}}_{K,M,N} \binom{m}{j} \frac{1}{m!} \frac{2^m \pi^{\frac{1}{2}m(m+1)}}{\prod_{i=1}^m \Gamma^2(i)} \times$$
$$\int_0^1 \cdots \int_0^1 \prod_{i_1 < i_2} |\sigma_{i_1}^2 - \sigma_{i_2}^2| \prod_{i=1}^{j} \sigma_i^2 \prod_{i=1}^{m} \sigma_i^L (1 - \sigma_i^2)^{\frac{l_M - m - 1}{2}} d\sigma_1 \cdots d\sigma_m \qquad (4.2.89)$$

as well as:

$$\langle \epsilon_j(AA^T)\rangle_{A_m}^{\text{IndSpherical}} = C_{M,m,n}^{\text{IndSpherical}} \binom{m}{j} \frac{1}{m!} \frac{2^m \pi^{\frac{1}{2}m(m+1)}}{\prod_{i=1}^m \Gamma^2(i)} \times$$
$$\int_0^\infty \cdots \int_0^\infty \prod_{i_1<i_2} |\sigma_{i_1}^2 - \sigma_{i_2}^2| \prod_{i=1}^j \sigma_i^2 \prod_{i=1}^m \frac{\sigma_i^L}{(1+\sigma_i^2)^{\frac{n+L+m}{2}}} d\sigma_1 \cdots d\sigma_m. \tag{4.2.90}$$

A simple change of variables reduces $\langle \epsilon_j(AA^T)\rangle_{A_m}^{\text{IndJacobi}}$ to the well-known Aomoto integral [Meh04], page 309, while another simple change of variables gives $\langle \epsilon_j(AA^T)\rangle_{A_m}^{\text{IndSpherical}}$. □

We have thus computed the characteristic average for the real induced Jacobi and spherical ensemble. Using theorem 4.1.10 we can relate these characteristic averages to the complex mean eigenvalue densities of both ensembles as follows,

Theorem 4.2.10. *(a) The mean density of complex eigenvalues for the real induced Jacobi ensemble is given by:*

$$\rho_{\mathit{IndJacobi},N}^{\mathbb{C}}(z) = \frac{2l_M(l_M-1)}{\pi} |\operatorname{Im}(z)|\, w_{\mathit{IndJacobi},1}^2(z) \times$$
$$\sum_{j=0}^{N-2} \frac{\Gamma(l_N+j+1)}{\Gamma(L+j+1)\Gamma(l_M+1)} |z|^{2j}. \tag{4.2.91}$$

(b) The mean density of complex eigenvalues for the real induced spherical ensemble is given by:

$$\rho_{\mathit{IndSpherical},N}^{\mathbb{C}}(z) = \frac{2}{\pi} |\operatorname{Im}(z)|\, w_{\mathit{IndSpherical},1}^2(z) \times$$
$$\sum_{j=0}^{N-2} \frac{\Gamma(n+L+1)}{\Gamma(L+j+1)\Gamma(n-j-1)} |z|^{2j}. \tag{4.2.92}$$

4.2.4 The $(K', L')-$correlation functions and the eigenvalue densities

In section 4.1 we formulated a strategy for determining the $(K', L')-$correlation functions and the eigenvalue densities for a particular real asymmetric random matrix ensemble. In this section we apply the same strategy to the real induced spherical and Jacobi ensemble. The idea is to use corollary 4.1.11 to determine the family of skew-symmetric polynomials necessary for the application of the method of skew-orthogonal polynomial, as outlined in section 4.1.2. Equipped with these skew-orthogonal polynomials the Pfaffian kernel of the $(K', L')-$correlation functions can be calculated, which in turns gives the eigenvalue densities.

In the following it will be helpful to define:

$$S^{\text{IndJacobi}}(j) := \frac{2}{\pi} \frac{\Gamma(l_N + j + 1)}{\Gamma(L + j + 1)\Gamma(l_M - 1)} \tag{4.2.93}$$

$$S^{\text{IndSpherical}}(j) := \frac{2}{\pi} \frac{\Gamma(n + L + 1)}{\Gamma(L + j + 1)\Gamma(n - j - 1)}. \tag{4.2.94}$$

Theorem 4.2.11. *(a) The following family of polynomials $\{q_j^{IndJacobi}\}_{j=0,1,\ldots}$ is skew-orthogonal with respect to the skew-inner product $(-,-)^{IndJacobi}$:*

$$q_{2j}^{IndJacobi}(z) = z^{2j}, \tag{4.2.95}$$

$$q_{2j+1}^{IndJacobi}(z) = z^{2j+1} - \frac{L + 2j}{l_N + 2j}\, z^{2j-1} \quad \text{for} \quad j = 1, 2, \ldots \tag{4.2.96}$$

with normalization constant:

$$r_j^{IndJacobi} := (q_{2j}^{IndJacobi}, q_{2j+1}^{IndJacobi})^{IndJacobi} = \pi\, \frac{\Gamma(l_M - 1)\Gamma(L + 2j + 1)}{\Gamma(l_N + 2j + 1)}.$$

Moreover $q_0^{IndJacobi}(z) = 1$ and $q_1^{IndJacobi}(z) = z$.

(b) The following family of polynomials $\{q_j^{IndSpherical}\}_{j=0,1,\ldots}$ is skew-orthogonal with respect to the skew-inner product $(-,-)^{IndSpherical}$:

$$q_{2j}^{IndSpherical}(z) = z^{2j}, \tag{4.2.97}$$

$$q_{2j+1}^{IndSpherical}(z) = z^{2j+1} - \frac{2j + L}{n - 2j - 1} z^{2j-1} \quad \text{for} \quad j = 1, 2, \ldots \tag{4.2.98}$$

with normalization constant:

$$r_j^{IndSpherical} := (q_{2j}^{IndSpherical}, q_{2j+1}^{2})^{IndSpherical} = \pi \frac{\Gamma(L + 2j + 1)\Gamma(n + N - 2j - 1)}{\Gamma(n + L + 1)}.$$

Moreover $q_0^{IndSpherical}(z) = 1$ and $q_1^{IndSpherical}(z) = z$.

Proof. Again corollary (4.1.11) is used in order to relate the characteristic average over the respective matrix measure to the entries of the Pfaffian kernel describing the eigenvalue correlation functions:

$$\begin{aligned} &\frac{c_{N,k,l}^I}{c_{N-2,k,l-1}^I}(z - v)\langle \det\left((A_{N-2} - zI_{N-2})(A_{N-2} - vI_{N-2})\right)\rangle_{A_{N-2}}^I \\ =&2 \sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_{2j}^I} \left[q_{2j}^I(z) q_{2j+1}^I(v) - q_{2j+1}^I(z) q_{2j}^I(v)\right]. \end{aligned} \tag{4.2.99}$$

Using theorem (4.2.9) then gives for $I \in \{\text{IndJacobi}, \text{IndSpherical}\}$:

$$(z-v)\sum_{j=0}^{N-2} S^I(j)(zv)^j = 2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r^I_{2j}} \left[q^I_{2j}(z)q^I_{2j+1}(v) - q^I_{2j+1}(z)q^I_{2j}(v)\right] \tag{4.2.100}$$

with $S^I(N-2,j)$ defined in equations (4.2.93)–(4.2.93). Now q^I_j is a monic polynomial of degree j. Hence it is possible to write:

$$\begin{aligned}
&\frac{1}{r^I_{N-2}}\left(q^I_{N-2}(z)q^I_{N-1}(v) - q^I_{N-1}(z)q^I_{N-2}(v)\right)\\
=&(z-v)\left(S^I(N-2)(zv)^{N-2} + S^I(N-3)(zv)^{N-3}\right)\\
=&S^I(N-2)z^{N-1}v^{N-2} - S^I(N-2)z^{N-2}v^{N-1}\\
&+ S^I(N-3)z^{N-1}v^{N-3} - S^I(N-3)z^{N-3}v^{N-2}.
\end{aligned} \tag{4.2.101}$$

Using $q^I_{2j}(z) = z^{2j}$ then yields:

$$\begin{aligned}
&\frac{1}{r^I_{N-2}}\left(q^I_{N-2}(z)q^I_{N-1}(v) - q^I_{N-1}(z)q^I_{N-2}(v)\right)\\
=&S^I(N-2)\Big[q^I_{N-2}(v)\Big(z^{N-1} - \frac{S^I(N-3)}{S^I(N-2)}z^{N-3}\Big)\\
&- q^I_{N-2}(z)\Big(v^{N-1} - \frac{S^I(N-3)}{S^I(N-2)}v^{N-3}\Big)\Big].
\end{aligned} \tag{4.2.102}$$

As a consequence:

$$q^I_{2j+1}(z) = z^{2j+1} - \frac{S^I(2j-1)}{S^I(2j)}z^{2j-1} \tag{4.2.103}$$

as well as $r^I_{2j} = 2S^I(2j)^{-1}$. □

Setting $L = 0$ in part (a) of theorem 4.2.11 we recover the skew-orthogonal polynomials found in [For10a] used in the context of truncations of random orthogonal matrices. Note that in general the polynomials, which are skew-orthogonal with respect to a certain weight function are not unique. As an example setting $L = 0$ in part (b) of theorem 4.2.11 does not yield the skew-orthogonal polynomials used in [FM11] in the context of the real spherical ensemble. Equipped with the necessary skew-orthogonal polynomials we can proceed to determining the Pfaffian kernel entries of the (K', L')-correlation functions. For this purpose we define:

$$s_N^{\text{IndJacobi}}(v,z) := \frac{2l_M(l_M-1)}{\pi} w_{\text{IndJacobi},1}(v) w_{\text{IndJacobi},1}(z) \sum_{j=0}^{N-2} \frac{\Gamma(l_N+j+1)}{\Gamma(L+j+1)\Gamma(l_M+1)}(vz)^j \tag{4.2.104}$$

$$s_N^{\text{IndSpherical}}(v,z) := \frac{2}{\pi} w_{\text{IndSpherical},1}(v) w_{\text{IndSpherical},1}(z) \sum_{j=0}^{N-2} \frac{\Gamma(n+N+1)}{\Gamma(L+j+1)\Gamma(n-j-1)}(vz)^j \tag{4.2.105}$$

$$r^{\text{IndJacobi}}(x,z) := \frac{1}{B\left(\frac{l_M}{2},\frac{M}{2}\right)} \operatorname{sgn}(x) z^{M-1} |1-z^2|^{\frac{l_M-2}{2}} I_{x^2}\left(\frac{M-1}{2},\frac{l_M}{2}\right) \tag{4.2.106}$$

$$r^{\text{IndSpherical}}(x,z) := \frac{1}{B\left(\frac{n-N+1}{2},\frac{M}{2}\right)} \operatorname{sgn}(x) z^{M-1} |1+z^2|^{-\frac{n+L+1}{2}} I_{\frac{x^2}{1+x^2}}\left(\frac{M+1}{2},\frac{n-N+2}{2}\right) \tag{4.2.107}$$

$$t^{\text{IndJacobi}}(x,z) := \frac{1}{B\left(\frac{l_M}{2},\frac{L+1}{2}\right)} z^L |1-z^2|^{\frac{l_M-2}{2}} I_{1-x^2}\left(\frac{l_M}{2},\frac{L}{2}\right) \tag{4.2.108}$$

$$t^{\text{IndSpherical}}(x,z) := \frac{1}{B\left(\frac{n}{2},\frac{L+1}{2}\right)} z^L |1+z^2|^{-\frac{n+L+1}{2}} I_{\frac{1}{1+x^2}}\left(\frac{n+1}{2},\frac{L}{2}\right) \tag{4.2.109}$$

Theorem 4.2.12. *Let $I \in \{IndJacobi, IndSpherical\}$ either denote the real induced Jacobi ensemble of $N\times N$ matrices with parameters K, M or the real induced spherical ensemble of $N \times N$ matrices with parameters n, M. Then the entries of the complex/complex (2×2) matrix kernel $K_N^I(z,w)$ in (4.1.20)–(4.1.21) are given by:*

$$\begin{aligned} DS_N^I(z,v) &= (v-z)s_N^I(z,v); \\ S_N^I(z,v) &= i(\bar{v}-z)s_N^I(z,\bar{v}); \\ IS_N^I(z,v) &= (\bar{z}-\bar{v})s_N^I(\bar{z},\bar{v}). \end{aligned}$$

The entries of the real/complex and complex/real matrix kernels $K_N^I(x,z)$ and $K_N^I(z,x)$ in (4.1.20)–(4.1.21) are given by:

$$\begin{aligned} DS_N^I(x,z) &= (z-x)s_N^I(x,z); \quad DS_N^{(2)}(z,x) = -DS_N^I(x,z); \\ S_N^I(x,z) &= i(\bar{z}-x)s_N^I(x,\bar{z}); \quad S_N(z,x) = s_N^I(x,z) + r^I(x,z) + t^I(x,z); \\ IS_N^I(x,z) &= -is_N^I(x,\bar{z}) - ir^I(x,\bar{z}) - it^I(x,\bar{z}); \quad IS_N^I(z,x) = -IS_N^I(x,z)\ . \end{aligned}$$

And finally, the entries of the real/real matrix kernel $K_N^I(x,y)$ in (4.1.20)–(4.1.30) *are given by:*

$$DS_N^I(x,y) = (y-x)s_N^I(x,y); \quad S_N^I(x,y) = s_N^I(x,y) + r^I(y,x) + t^I(x,y);$$
$$IS_N^I(x,y) = -\int_x^y S_N^I(t,y)dt$$

Using equation (4.1.34) we can relate the entries of the Pfaffian correlation kernel to the mean density of real eigenvalues. As a result:

Corollary 4.2.13. *(a) The mean density of real eigenvalues of a real induced Jacobi matrix is given by:*

$$\begin{aligned}\rho^{\mathbb{R}}_{IndJacobi,N}(x) &= \frac{1}{B\left(\frac{l_M}{2},\frac{1}{2}\right)} x^{2L}(1-x^2)^{l_M-1}\sum_{j=0}^{N-2}\frac{\Gamma(l_N+j)}{\Gamma(l_M)\Gamma(L+j+1)}x^{2j}\\ &+ \frac{1}{B\left(\frac{l_M}{2},\frac{L+1}{2}\right)}x^L(1-x^2)^{\frac{l_M-2}{2}} I_{1-x^2}\Big(\frac{l_M}{2},\frac{L}{2}\Big)\\ &+ \frac{1}{B\left(\frac{l_M}{2},\frac{M}{2}\right)}|x|^{M-1}(1-x^2)^{\frac{l_M-2}{2}} I_{x^2}\Big(\frac{M-1}{2},\frac{l_M}{2}\Big). \end{aligned} \tag{4.2.110}$$

(b) The mean density of real eigenvalues of a real induced spherical matrix is given by:

$$\begin{aligned}\rho^{\mathbb{R}}_{IndSpherical,N}(x) &= \frac{1}{B\left(\frac{n+L}{2},\frac{1}{2}\right)}\frac{x^{2L}}{(1+x^2)^{n+L}}\sum_{j=0}^{N-2}\frac{\Gamma(n+L)}{\Gamma(n-j)\Gamma(L+j+1)}x^{2j}\\ &+ \frac{1}{B\left(\frac{n}{2},\frac{L+1}{2}\right)}\frac{x^L}{(1+x^2)^{\frac{n+L+1}{2}}} I_{\frac{1}{1+x^2}}\Big(\frac{n+1}{2},\frac{L}{2}\Big)\\ &+ \frac{1}{B\left(\frac{n-N+1}{2},\frac{M}{2}\right)}\frac{|x|^{M-1}}{(1+x^2)^{\frac{n+L+1}{2}}} I_{\frac{x^2}{1+x^2}}\Big(\frac{M+1}{2},\frac{n-N+2}{2}\Big). \end{aligned} \tag{4.2.111}$$

4.2.5 Asymptotic analysis: The real induced spherical ensemble

In the following section the asymptotic behavior of the eigenvalue statistics of the real induced spherical ensemble is described. As in the complex case four distinct asymptotic regimes can be distinguished depending on the parameters of the real induced spherical ensemble.

The main distinction between the different asymptotic regime is the support of the limiting eigenvalue density. However after an inverse stereographical projection to the unit sphere the eigenvalues are either uniformly distributed on a so-called spherical annulus or on the entire sphere. In all four regimes, in the bulk of the eigenvalue support the limiting correlation kernel shows universal behavior. More precisely the correlation kernels of the real induced Ginibre ensemble (from section B.1 are found in the respective asymptotic regimes. Starting point of the asymptotic analysis are the integral representations of the mean eigenvalue densities.

The mean density of complex eigenvalues for the real induced spherical ensemble is given by:

$$\rho^{\mathbb{C}}_{\text{IndSpherical},N}(z) = \frac{(n+L)(n+L+1)}{\pi}|\operatorname{Im}(z)|\frac{|1+|z|^2|^{n+L-2}}{|1+z^2|^{n+L+1}}\times$$
$$\int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty}(1+u^2)^{-\frac{n+L+1}{2}}du\left[I_{\frac{|z|^2}{1+|z|^2}}(L,n-1)-I_{\frac{|z|^2}{1+|z|^2}}(M-1,n-N)\right]. \quad (4.2.112)$$

The mean density of real eigenvalues for the real induced spherical ensemble can be written as:

$$\begin{aligned}\rho^{\mathbb{R}}_{\text{IndSpherical},N}(x) = &\frac{1}{B\left(\frac{n+L}{2},\frac{1}{2}\right)}\frac{1}{1+x^2}\left[I_{\frac{x^2}{1+x^2}}(L,n)-I_{\frac{x^2}{1+x^2}}(M,n-N+1)\right]\\ &+\frac{1}{B\left(\frac{n-N+1}{2},\frac{M}{2}\right)}\frac{|x|^{M-1}}{(1+x^2)^{\frac{n+L+1}{2}}}I_{\frac{x^2}{1+x^2}}\left(\frac{M+1}{2},\frac{n-N+2}{2}\right)\\ &+\frac{1}{B\left(\frac{n}{2},\frac{L+1}{2}\right)}\frac{x^L}{(1+x^2)^{\frac{n+L+1}{2}}}I_{\frac{1}{1+x^2}}\left(\frac{n+1}{2},\frac{L}{2}\right). \quad (4.2.113)\end{aligned}$$

Figure 4.2 shows the eigenvalue distribution of the real induced spherical ensemble in the four asymptotic regimes, while figure 4.3 shows the eigenvalue distribution of the real induced spherical ensemble after an inverse stereographical projection to the unit sphere.

Strong rectangularity and strong spherical component

In the regime of strong rectangularity with strong spherical component the rectangularity parameter as well as the spherical component are scaled proportional to matrix size $L = N\alpha$ and $n-N = N\beta$. In addition set $\frac{L}{n} := \mu_1$ and $\frac{M}{n-N} =: \mu_2$.

In the limit of large matrix dimensions the mean density of complex eigenvalues is supported on an annulus of width $\sqrt{\mu_2}-\sqrt{\mu_1}$. Thus the density has two cut-offs, the inner edge with radius $r^{\text{in}} = \sqrt{\mu_1}$ and the outer edge with radius $r^{\text{out}} = \sqrt{\mu_2}$, while the mean density of real eigenvalues is supported on two disjoint intervals: $[-\sqrt{\mu_2},-\sqrt{\mu_1}]$ and $[\sqrt{\mu_1},\sqrt{\mu_2}]$ and the number of real eigenvalues is:

$$\mathcal{N}^R_{\text{IndSpherical}} \sim \sqrt{\frac{2(n+L)}{\pi}}\big(\arctan(\sqrt{\mu_2}) - \arctan(\sqrt{\mu_1})\big). \tag{4.2.114}$$

Thus the number of real eigenvalues is of order $\sqrt{N}$. Close to the edges of the eigenvalue support the mean eigenvalue densities exhibit universal behavior of Feinberg-Zee type for $\beta = 1$. Similarly closing down on the real line $z = x + iy$ with $y = \frac{u}{\sqrt{n+L}}$ the mean density of complex eigenvalues becomes:

$$\lim_{N\to\infty} \frac{1}{n+L}\rho^{\mathbb{C}}_{\text{IndSpherical}}(z) = \big(\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\big)^2 h\big(u\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\big), \tag{4.2.115}$$

where

$$h(u) = 4\pi|u|\,\mathrm{e}^{4\pi u^2}\,\mathrm{erfc}(\sqrt{4\pi}|u|) \tag{4.2.116}$$

$$\rho^{\mathbb{R}}_{\text{IndSpherical}}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{n+L}}\rho^{\mathbb{R}}_{\text{IndSpherical},N}(z). \tag{4.2.117}$$

Furthermore in the limit of large matrix dimensions after unfolding the respective correlation kernels one recovers the same limiting expressions as in the case of the real Ginibre ensemble. A detailed account of the limiting expression for the correlation kernel in the bulk, can be found in section B.2, see theorem B.2.1. Moreover,

Theorem 4.2.14. *In the regime of strong rectangularity and strong spherical component, $L = N\alpha$ and $n - N = N\beta$ in the limit of large matrix dimension the mean density of complex eigenvalues is given by:*

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{n+L}\ \rho^{\mathbb{C}}_{\mathit{IndSpherical},N}(z)\\ =&\frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\ \big[\Theta\big(|z| - \sqrt{\mu_1}\ \big) - \Theta\big(|z| - \sqrt{\mu_2}\ \big)\big] =: \rho^{\mathbb{C}}_{\mathit{IndSpherical}}(z)\ ,\end{aligned} \tag{4.2.118}$$

while the mean density of real eigenvalues is given by:

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{\sqrt{n+L}}\ \rho^{\mathbb{R}}_{\mathit{IndSpherical},N}(z)\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\ \big[\Theta\big(|x| - \sqrt{\mu_1}\ \big) - \Theta\big(|x| - \sqrt{\mu_2}\ \big)\big] =: \rho^{\mathbb{R}}_{\mathit{IndSpherical}}(z).\end{aligned} \tag{4.2.119}$$

At the edges of $z^{in} = \big(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\ \big)\,\mathrm{e}^{i\phi}$ and $z^{out} = \big(\sqrt{\mu_2} + \frac{\xi}{\sqrt{n+L}}\ \big)\,\mathrm{e}^{i\phi}$ of the *complex eigenvalue density:*

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(z^{in}\big)=\frac{1}{2\pi}\frac{1}{(1+\mu_1)^2}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big) \tag{4.2.120}$$
$$=\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\operatorname{erfc}\Big(\sqrt{2\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_1}\big)}\xi\Big)$$
$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(z^{out}\big)=\frac{1}{2\pi}\frac{1}{(1+\mu_2)^2}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\Big) \tag{4.2.121}$$
$$=\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_2}\big)\frac{1}{2\pi}\operatorname{erfc}\Big(\sqrt{2\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_2}\big)}\xi\Big).$$

At the edges $x^{in}=\sqrt{\mu_1}-\frac{\xi}{\sqrt{n+L}}$ and $x^{out}=\sqrt{\mu_2}+\frac{\xi}{\sqrt{n+L}}$ of the real eigenvalue density:

$$\lim_{N\to\infty}\frac{1}{\sqrt{n+L}}\rho^{\mathbb{R}}_{IndSpherical,N}(x^{in}) \tag{4.2.122}$$
$$=\frac{1}{\sqrt{2\pi}}\Big[\frac{1}{1+\mu_1}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big)+\frac{1}{2\sqrt{2}}\frac{1}{1+\mu_1}e^{\frac{u^2}{1+\mu_1}}\operatorname{erfc}\Big(-\frac{1}{\sqrt{1+\mu_1}}\xi\Big)\Big]$$
$$\lim_{N\to\infty}\frac{1}{\sqrt{n+L}}\rho^{\mathbb{R}}_{IndSpherical,N}\big(x^{out}\big) \tag{4.2.123}$$
$$=\frac{1}{2\pi}\Big[\frac{1}{1+\mu_2}\operatorname{erfc}\big(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\big)+\frac{1}{2\sqrt{2}}\frac{1}{1+\mu_2}e^{\frac{u^2}{1+\mu_2}}\operatorname{erfc}\Big(-\frac{1}{\sqrt{1+\mu_2}}\xi\Big)\Big].$$

Closing down on the real line with scaling $z=x+iy$ and $y=\frac{u}{\sqrt{n+L}}$ the complex eigenvalue density becomes:

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(x+i\frac{u}{\sqrt{n+L}}\big)=\sqrt{\frac{2}{\pi}}|u|\frac{1}{(1+x^2)^3}\,\mathrm{e}^{\frac{2v^2}{1+x^2}}\operatorname{erfc}\big(\sqrt{2}\frac{|u|}{1+x^2}\big). \tag{4.2.124}$$

Proof. As in the case of the complex induced spherical ensemble, we know:

$$\lim_{N\to\infty}I_{\frac{|z|^2}{1+|z|^2}}(N\alpha,N(\beta+1)-1)=\Theta\Big(|z|-\sqrt{\mu_1}\Big) \tag{4.2.125}$$
$$\lim_{N\to\infty}I_{\frac{|z|^2}{1+|z|^2}}(N(\alpha+1)-1,N\beta)=\Theta\Big(|z|-\sqrt{\mu_2}\Big) \tag{4.2.126}$$

Furthermore note with $z = x + iy$ and $b = 1 - \frac{4y^2}{(1+x^2+y^2)}$:

$$\begin{aligned}
&2|y| \frac{(1+|z|^2)^{n+L-2}}{|1+z^2|^{n+L+1}} \int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty} (1+u^2)^{-\frac{n+L+1}{2}} du \\
&= \frac{1}{2} \frac{1}{(1+|z|^2)^2} \frac{(1+x^2+y^2)^{n+L}}{\big((1+x^2+y^2)^2 - 4y^2\big)^{\frac{n+L}{2}}} \frac{2|y|}{|1+z^2|} \times \\
&\int_{\frac{4y^2}{(1+x^2+y^2)^2-4y^2)}}^{\infty} u^{-\frac{1}{2}} (1+u)^{-\frac{n+L+1}{2}} du \\
&= \frac{1}{2} \frac{1}{(1+|z|^2)^2} b^{-\frac{n+L}{2}} \Big(\frac{4y^2}{(1+x^2+y^2)^2 - 4y^2} \Big)^{\frac{1}{2}} \int_{\frac{4y^2}{(1+x^2+y^2)^2-4y^2}}^{\infty} u^{-\frac{1}{2}} (1+u)^{-\frac{n+L+1}{2}} du \\
&= \frac{1}{2} \frac{1}{(1+|z|^2)^2} \, b^{\frac{n+L}{2}} (b^{-1}-1)^{\frac{1}{2}} \int_{b^{-1}-1}^{\infty} u^{-\frac{1}{2}} (1+u)^{\frac{n+L}{2}} du \qquad (4.2.127)
\end{aligned}$$

Now change variables $w = \log(\frac{b}{u})$ and apply Watsons lemma, then:

$$2|y| \frac{(1+|z|^2)^{n+L-2}}{|1+z^2|^{n+L+1}} \int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty} (1+u^2)^{-\frac{n+L+1}{2}} du \sim \frac{1}{(1+|z|^2)^2} \frac{1}{n+L}, \qquad (4.2.128)$$

which proves equation (4.2.118). The edge asymptotics for the mean density of complex eigenvalue then follow analogously by applying theorem A.2.4. In the vicinity of the real line $z = x + iy$ we note that:

$$\begin{aligned}
|1+z^2|^{n+L+1} =& \Big(\big(1 + x^2 + \frac{v^2}{n+L}\big)^2 - 4 \frac{v^2}{n+L} \Big)^{\frac{n+L+1}{2}} \\
=& \Big((1+x^2)^2 + 2 \frac{v^2}{n+L} (1+x^2) + \frac{v^4}{(n+L)^2} - 4 \frac{v^2}{n+L} \Big)^{\frac{n+L+1}{2}} \\
=& (1+x^2)^{n+L+1} \Big(1 + 2 \frac{v^2}{n+L} \frac{x^2-1}{(1+x^2)^2} + \frac{v^4}{(n+L)^2 (1+x^2)^2} \Big)^{\frac{n+L+1}{2}} \\
\sim& (1+x^2)^{n+L+1} \, \mathrm{e}^{v^2 \frac{1-x^2}{(1+x^2)^2}} .
\end{aligned}$$

As well as:

$$|1+|z|^2|^{n+L-2} \sim (1+x^2)^{n+L+1} \, \mathrm{e}^{v^2 \frac{1}{1+x^2}} . \qquad (4.2.129)$$

Furthermore note that:

$$\frac{2|y|}{|1+z^2|} \sim \frac{2|v|}{\sqrt{n+L}} \frac{1}{1+x^2}. \qquad (4.2.130)$$

In addition to that for large N:

$$\begin{aligned}
&\int_{\frac{a}{\sqrt{N}}}^{\infty} \frac{1}{(1+t^2)^N} dt = \int_0^{\infty} \frac{1}{(1+t^2)^N} dt - \int_0^{\frac{a}{\sqrt{N}}} \frac{1}{(1+t^2)^N} dt \\
=&\int_0^1 (1-t^2)^{N-2} dt - \frac{1}{\sqrt{N}} \int_0^a \frac{1}{(1+\frac{t^2}{N})^N} dt \\
\sim& B\big(\frac{1}{2}, N-1\big) - \frac{1}{\sqrt{N}} \int_0^a \mathrm{e}^{-t^2}\, dt \\
\sim& \frac{1}{2}\sqrt{\frac{\pi}{N}} - \frac{1}{\sqrt{N}} \int_0^a \frac{1}{(1+\frac{t^2}{N})^N} dt \sim \frac{1}{2}\sqrt{\frac{\pi}{N}}\,\mathrm{erfc}(a).
\end{aligned} \tag{4.2.131}$$

As a result:

$$\int_{\frac{2|y|}{|1+z^2|}}^{\infty} \frac{1}{(1+t^2)^{\frac{n+L+1}{2}}} dt \sim \sqrt{\frac{\pi}{2(n+L)}}\,\mathrm{erfc}(a). \tag{4.2.132}$$

All together we derived equation (4.2.182). At the outer real edge first note that theorems A.2.1 and A.2.5 yield:

$$I_{\frac{(x^{\text{out}})^2}{1+(x^{\text{out}})^2}}(L,n) \sim 1 - I_{\frac{(x^{\text{out}})^2}{1+(x^{\text{out}})^2}}\big(\frac{n+1}{2},\frac{L}{2}\big) \sim 1 \tag{4.2.133}$$

$$I_{\frac{(x^{\text{out}})^2}{1+(x^{\text{out}})^2}}(M, n-N+1) \sim \frac{1}{2} + \frac{1}{2}\,\mathrm{erfc}\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big) \tag{4.2.134}$$

$$I_{\frac{(x^{\text{out}})^2}{1+(x^{\text{out}})^2}}\big(\frac{M+1}{2}, \frac{n-N+2}{2}\big) \sim \frac{1}{2} + \frac{1}{2}\,\mathrm{erfc}\Big(\frac{1}{\sqrt{1+\mu_1}}\xi\Big). \tag{4.2.135}$$

Furthermore note that:

$$\frac{1}{B\big(\frac{n-N+1}{2},\frac{M}{2}\big)} \sim \frac{1}{\sqrt{4\pi}}\Big(\frac{n+L+1}{n-N+1}\Big)^{\frac{n-N}{2}} \sqrt{M}\Big(\frac{n+L+1}{M}\Big)^{\frac{n_N}{2}}, \tag{4.2.136}$$

as well as:

$$\begin{aligned}
|x^{\text{out}}|^{M-1} =& \Big(\frac{M}{n-N+1}\Big)^{\frac{M-1}{2}} \big(1 + \frac{2u}{\sqrt{n+L}}\sqrt{\frac{n-N+1}{M}} + \frac{u^2}{n+L}\frac{n-N+1}{M}\big)^{\frac{M-1}{2}} \\
\sim& \Big(\frac{M}{n-N+1}\Big)^{\frac{M-1}{2}} \mathrm{e}^{-\frac{1}{2}\frac{u^2}{1+\mu_2}}.
\end{aligned} \tag{4.2.137}$$

In addition to that:

$$|1+(x^{\text{out}})^2|^{-\frac{n+L+1}{2}} \sim \Big(\frac{n-N+1}{n+L+1}\Big)^{\frac{n+L+1}{2}} \mathrm{e}^{-\frac{1}{2}\frac{u^2}{1+\mu_2}}. \tag{4.2.138}$$

All in all:

$$\frac{1}{B\left(\frac{n-N+1}{2},\frac{M}{2}\right)}\frac{|x^{\text{out}}|^{M-1}}{|1+(x^{\text{out}})^2|^{-\frac{n+L+1}{2}}}I_{\frac{(x^{\text{out}})^2}{1+(x^{\text{out}})^2}}\left(\frac{M+1}{2},\frac{n-N+2}{2}\right)$$
$$\sim\frac{1}{4\sqrt{\pi}}\frac{1}{1+\mu_2}\operatorname{erfc}\Big(-\frac{1}{\sqrt{1+\mu_1}}\xi\Big). \tag{4.2.139}$$

Similarly we show that:

$$\lim_{N\to\infty}\frac{1}{B\left(\frac{L+1}{2},\frac{n}{2}\right)}\frac{|x^{\text{out}}|^{L}}{|1+(x^{\text{out}})^2|^{-\frac{n+L+1}{2}}}I_{\frac{1}{1+(x^{\text{out}})^2}}\left(\frac{n+1}{2},\frac{L}{2}\right)=0, \tag{4.2.140}$$

which proves the limiting form of the real eigenvalue density at the outer edge. The inner edge calculation follows analogously. $\square$

Strong rectangularity and weak spherical component

In the regime of strong rectangularity with weak spherical component regime the rectangularity parameter is again scaled proportional to matrix size $L = N\alpha$, while the spherical component $n - N = O(1)$ is kept fixed. Furthermore set $\frac{L}{n-1} \sim \alpha =: \mu_1$.

In this regime in the limit of large matrix dimensions the mean density of complex eigenvalues is supported on the whole complex plane except on a disk around the origin with radius $\sqrt{\mu_1}$. Thus the density possesses an inner circular edge with radius $r^{\text{in}} = \sqrt{\mu_1}$. The mean density of real eigenvalues is supported on two disjoint intervals: $(-\infty, -\sqrt{\mu_1}\,]$ and $[\sqrt{\mu_1}, \infty)$ and the number of real eigenvalues is:

$$\mathcal{N}^{R}_{\text{IndSpherical}} \sim \sqrt{\frac{2M}{\pi}}\Big(\frac{\pi}{2}-\arctan(\sqrt{\mu_1})\Big). \tag{4.2.141}$$

Thus the number of real eigenvalues is of order $\sqrt{N}$. Close to the edges of the eigenvalue support the mean eigenvalue densities exhibit universal behavior of Feinberg-Zee type for $\beta = 1$. Similarly closing down on the real line $z = x + iy$ with $y = \frac{u}{\sqrt{n+L}}$ the mean density of complex eigenvalues becomes:

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{\text{IndSpherical}}(z) = \big(\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\big)^2 h\big(u\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\big), \tag{4.2.142}$$

where

$$h(u) = 4\pi|u|\,\mathrm{e}^{4\pi u^2}\,\mathrm{erfc}(\sqrt{4\pi}|u|) \tag{4.2.143}$$

$$\rho^{\mathbb{R}}_{\mathrm{IndSpherical}}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{n+L}} \rho^{\mathbb{R}}_{\mathrm{IndSpherical},N}(z). \tag{4.2.144}$$

Furthermore in the limit of large matrix dimensions after unfolding the respective correlation kernels one recovers the same limiting expressions as in the case of the real Ginibre ensemble. A detailed account of the limiting expression for the correlation kernel in the bulk, can be found in section B.2, see theorem B.2.2. Moreover:

Theorem 4.2.15. *In the regime of strong rectangularity with weak spherical component, $L = N\alpha$ and $n - N = O(1)$, in the limit of large matrix dimension, the mean density of complex eigenvalues is given by:*

$$\lim_{N\to\infty} \frac{1}{n+L}\, \rho^{\mathbb{C}}_{\mathit{IndSpherical},N}(z) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\,\Theta\big(|z| - \sqrt{\mu_1}\,\big) =: \rho^{\mathbb{C}}_{\mathit{IndSpherical}}(z), \tag{4.2.145}$$

while the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty} \frac{1}{\sqrt{n+L}}\, \rho^{\mathbb{R}}_{\mathit{IndSpherical},N}(z) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\,\Theta\big(|x| - \sqrt{\mu_1}\,\big) =: \rho^{\mathbb{R}}_{\mathit{IndSpherical}}(z) \tag{4.2.146}$$

At the edge $z^{in} = \big(\sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}\big)\,\mathrm{e}^{i\phi}$ of the complex eigenvalue density:

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{n+L}\rho^{\mathbb{C}}_{\mathit{IndSpherical},N}\big(z^{in}\big) = \frac{1}{2\pi}\frac{1}{(1+\mu_1)^2}\,\mathrm{erfc}\,\big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\big)\\ =&\pi\rho^{\mathbb{C}}_{\mathit{IndSpherical}}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\,\mathrm{erfc}\,\Big(\sqrt{2\pi\rho^{\mathbb{C}}_{\mathit{IndSpherical}}\big(\sqrt{\mu_1}\big)}\xi\Big).\end{aligned} \tag{4.2.147}$$

At the inner edge $x^{in} = \sqrt{\mu_1} - \frac{\xi}{\sqrt{n+L}}$ of the real eigenvalue density:

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{\sqrt{n+L}}\, \rho^{\mathbb{R}}_{\mathit{IndSpherical},N}\big(x^{in}\big)\\ =&\frac{1}{\sqrt{2\pi}}\Big[\frac{1}{1+\mu_1}\,\mathrm{erfc}\,\Big(\frac{\sqrt{2}}{\sqrt{1+\mu_1}}\xi\Big) + \frac{1}{2\sqrt{2}}\frac{1}{1+\mu_1}e^{\frac{u^2}{1+\mu_1}}\,\mathrm{erfc}\,\Big(-\frac{1}{\sqrt{1+\mu_1}}\xi\Big)\Big].\end{aligned} \tag{4.2.148}$$

Closing down on the real line with scaling $z = x + iy$ and $y = \frac{u}{\sqrt{n+L}}$ the complex *eigenvalue density becomes:*

$$\lim_{N\to\infty}\frac{1}{n+L}\,\rho^{\mathbb{C}}_{\mathit{IndSpherical},N}\big(x+i\frac{u}{\sqrt{n+L}}\big)=\sqrt{\frac{2}{\pi}}|u|\frac{1}{(1+x^2)^3}\,\mathrm{e}^{\frac{2u^2}{1+x^2}}\,\mathrm{erfc}\,\big(\sqrt{2}\frac{|u|}{1+x^2}\big). \tag{4.2.149}$$

Proof. As in the case of the complex induced spherical ensemble applying A.2.1 yields:

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(L,n)=\Theta\big(|z|-\sqrt{\mu_1}\big) \tag{4.2.150}$$

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(M-1,N-n)=0. \tag{4.2.151}$$

Together with the proof of 4.2.14, this gives 4.2.145. The edge asymptotics for the mean density of complex eigenvalue then follow analogously by applying A.2.4. For the density of real eigenvalues we note that:

$$\lim_{N\to\infty} I_{\frac{x^2}{1+x^2}}\Big(\frac{n+1}{2},\frac{L}{2}\Big)=\Theta\big(|z|-\sqrt{\mu_1}\big) \tag{4.2.152}$$

$$\lim_{N\to\infty} I_{\frac{x^2}{1+x^2}}\Big(\frac{M+1}{2},\frac{N-n+2}{2}\Big)=0. \tag{4.2.153}$$

In addition to that:

$$\frac{1}{B\big(\frac{n-N+1}{2},\frac{M}{2}\big)}\frac{|x^{\mathrm{out}}|^{M-1}}{|1+(x^{\mathrm{out}})^2|^{\frac{n+L+1}{2}}} \tag{4.2.154}$$

$$\sim\frac{1}{\Gamma\big(\frac{n-N+1}{2}\big)}\,\mathrm{e}^{\frac{n-N+1}{2}}\,\Big(\frac{n+L+1}{2}\Big)^{\frac{n-N+1}{2}}\,\frac{|x^{\mathrm{out}}|^{M-1}}{|1+(x^{\mathrm{out}})^2|^{\frac{n+L+1}{2}}}. \tag{4.2.155}$$

Noting that:

$$0\le\frac{|x^{\mathrm{out}}|^{M-1}}{|1+(x^{\mathrm{out}})^2|}\le 1, \tag{4.2.156}$$

it follows that:

$$\lim_{N\to\infty}\frac{1}{B\big(\frac{n-N+1}{2},\frac{M}{2}\big)}\frac{|x^{\mathrm{out}}|^{M-1}}{|1+(x^{\mathrm{out}})^2|^{\frac{n+L+1}{2}}}I_{\frac{x^2}{1+x^2}}\Big(\frac{M+1}{2},\frac{N-n+2}{2}\Big)=0. \tag{4.2.157}$$

Similarly we can show:

$$\lim_{N\to\infty}\frac{1}{B\big(\frac{L+1}{2},\frac{n}{2}\big)}\frac{|x^{\mathrm{out}}|^{L}}{|1+(x^{\mathrm{out}})^2|^{\frac{n+L+1}{2}}}I_{\frac{1}{1+x^2}}\Big(\frac{n+1}{2},\frac{L}{2}\Big)=0, \tag{4.2.158}$$

which in turn proves the limiting form of the real mean eigenvalue density in the bulk. At the inner edge of the real eigenvalue density we again apply theorem A.2.5 and furthermore use proof of theorem 4.2.14. □

Almost square and strong spherical component

The rectangularity parameter $L = O(1)$ is kept fixed, while the spherical component grows proportionally to matrix size $n - N = N\beta$. Set $\frac{N+L-1}{N\beta} \sim \frac{1}{\beta} := \mu_2$.

In the regime of almost square matrices with strong spherical component the mean density of complex eigenvalues is supported on a ring around the origin with radius $r^{\text{out}} = \sqrt{\mu_2}$. Consequently the density possesses one outer circular edge, at which the density falls to zero at Gaussian rate and exibits universal behavior. The mean density of real eigenvalue is supported on the interval $[-\sqrt{\mu_2}, \sqrt{\mu_2}\ ]$ and shows universal behavior of the Feinberg-Zee type for $\beta = 1$ at the edge of its support. The average number of real eigenvalues is to leading order given by:

$$\mathcal{N}^{R}_{\text{IndSpherical}} \sim \sqrt{\frac{2n}{\pi}} \arctan\left(\sqrt{\mu_2}\right) \tag{4.2.159}$$

and thus is of order $\sqrt{N}$. Similarly closing down on the real line $z = x + iy$ with $y = \frac{u}{\sqrt{n+L}}$ the mean density of complex eigenvalues becomes:

$$\lim_{N\to\infty} \frac{1}{n+L} \rho^{\mathbb{C}}_{\text{IndSpherical}}(z) = \left(\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\right)^2 h\left(u\rho^{\mathbb{R}}_{\text{IndSpherical}}(x)\right), \tag{4.2.160}$$

where

$$h(u) = 4\pi|u|\, \mathrm{e}^{4\pi u^2} \operatorname{erfc}(\sqrt{4\pi}|u|) \tag{4.2.161}$$

$$\rho^{\mathbb{R}}_{\text{IndSpherical}}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{n+L}} \rho^{\mathbb{R}}_{\text{IndSpherical},N}(z). \tag{4.2.162}$$

Furthermore in the limit of large matrix dimensions after unfolding the respective correlation kernels in the bulk of the density one recovers the same limiting expressions as in the case of the real Ginibre ensemble. At the origin the correlation kernels corresponding to the real induced Ginibre ensembles are found. A detailed account of the limiting expression for the correlation kernel in the bulk and at the origin can be found in section B.2, see theorem B.2.3. More precisely,

Theorem 4.2.16. *In the regime of almost square with strong spherical component, $L = O(1)$ and $n - N = N\beta$ in the limit of large matrix dimension the mean density of complex eigenvalues is given by:*

$$\lim_{N\to\infty} \frac{1}{n+L} \rho^{\mathbb{C}}_{\mathit{IndSpherical},N}(z) = \frac{1}{\pi} \frac{1}{(1+|z|^2)^2} \Theta(\sqrt{\mu_2} - |z|) =: \rho^{\mathbb{C}}_{\mathit{IndSpherical}}(z), \tag{4.2.163}$$

while the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty}\frac{1}{\sqrt{n+L}}\rho^{\mathbb{R}}_{IndSpherical,N}(z)=\frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\Theta(\sqrt{\mu_2}-|x|)=:\rho^{\mathbb{R}}_{IndSpherical}(z). \tag{4.2.164}$$

At the edge $z^{out}=\left(\sqrt{\mu_2}+\frac{\xi}{\sqrt{n+L}}\right)\mathrm{e}^{i\phi}$ of the complex eigenvalue density:

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(z^{out}\big)&=\frac{1}{2\pi}\frac{1}{(1+\mu_2)^2}\,\mathrm{erfc}\left(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\right)\\ =&\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_2}\,\big)\frac{1}{2\pi}\,\mathrm{erfc}\left(\sqrt{2\pi\rho^{\mathbb{C}}_{IndSpherical}\big(\sqrt{\mu_2}\big)}\xi\right).\end{aligned} \tag{4.2.165}$$

At the inner edge $x^{out}=\sqrt{\mu_2}+\frac{\xi}{\sqrt{n+L}}$ of the real eigenvalue density:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{\sqrt{n+L}}\,\rho^{\mathbb{R}}_{IndSpherical,N}\big(x^{out}\big) \qquad (4.2.166)\\ =&\frac{1}{\sqrt{2\pi}}\Big[\frac{1}{1+\mu_2}\,\mathrm{erfc}\left(\frac{\sqrt{2}}{\sqrt{1+\mu_2}}\xi\right)+\frac{1}{2\sqrt{2}}\frac{1}{1+\mu_2}e^{\frac{u^2}{1+\mu_2}}\,\mathrm{erfc}\left(-\frac{1}{\sqrt{1+\mu_2}}\xi\right)\Big].\end{aligned}$$

Closing down on the real line with scaling $z=x+iy$ and $y=\frac{u}{\sqrt{n+L}}$ the complex eigenvalue density becomes:

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(x+i\frac{u}{\sqrt{n+L}}\big)=\sqrt{\frac{2}{\pi}}|u|\frac{1}{(1+x^2)^3}\,\mathrm{e}^{\frac{2u^2}{1+x^2}}\,\mathrm{erfc}\big(\sqrt{2}\frac{|u|}{1+x^2}\big). \tag{4.2.167}$$

Almost square and weak spherical component

In the regime of almost square matrices with weak spherical component both parameters are kept fixed $L, n-N=O(1)$. As a result in the limit of large matrix dimensions the mean eigenvalue density is supported on the whole complex plane and the eigenvalues are standard Chauchy distributed. Furthermore the average number of real eigenvalues is to leading order given by: $\sqrt{\frac{2N}{\pi}}$ as in the real Ginibre ensemble. Again closing down on the real line one recovers universal behavior for the mean density of complex eigenvalues. In the bulk regime the correlation kernels again show universal behavior, recovering results from [FM11]. At the origin the correlation kernels of the real induced Ginibre ensemble is recovered. Again a detailed derivation of the correlation kernel asymptotics is found in section B.2, see theorem B.2.4. We summarize,

Theorem 4.2.17. *In the regime of almost square with weak spherical component, $L=O(1)$ and $n-N=O(1)$ in the limit of large matrix dimension the mean* *density of complex eigenvalues is given by:*

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}(z)=\frac{1}{\pi}\frac{1}{(1+|z|^2)^2}=:\rho^{\mathbb{C}}_{IndSpherical}(z), \tag{4.2.168}$$

while the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty}\frac{1}{\sqrt{n+L}}\rho^{\mathbb{R}}_{IndSpherical,N}(z)=\frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}=:\rho^{\mathbb{R}}_{IndSpherical}(z). \tag{4.2.169}$$

Closing down on the real line with scaling $z=x+iy$ and $y=\frac{u}{\sqrt{n+L}}$ the complex eigenvalue density becomes:

$$\lim_{N\to\infty}\frac{1}{n+L}\rho^{\mathbb{C}}_{IndSpherical,N}\big(x+i\frac{u}{\sqrt{n+L}}\big)=\sqrt{\frac{2}{\pi}}|u|\frac{1}{(1+x^2)^3}\,\mathrm{e}^{\frac{2u^2}{1+x^2}}\operatorname{erfc}\big(\sqrt{2}\frac{|u|}{1+x^2}\big). \tag{4.2.170}$$

4.2.6 Asymptotic analysis: The real induced Jacobi ensemble

In the following section the asymptotic behavior of the real induced Jacobi ensemble is analyzed. As in the case of the complex induced Jacobi ensemble we can distinguish four different asymptotic regimes depending on the parameters L and l_M. Again in the regime of strong rectangularity and partially weak non-orthogonality the correlation functions show new behavior, while in the regimes of strong rectangularity and strong non-orthogonality and almost square matrices and strong non-orthogonality the universal Ginibre correlation kernel are recovered after appropriate unfolding. Finally in the regime of almost square matrices and weak non-orthogonality the correlation kernel of truncations of random orthogonal matrices in the regime of weak non-orthogonality is recovered. Starting point for the asymptotic analysis are again the integral representations of the mean eigenvalue densities. The mean density of complex eigenvalues for the real induced Jacobi ensemble is given by:

$$\begin{aligned}\rho^{\mathbb{C}}_{\mathrm{IndJacobi},N}(z)=\frac{2l_M(l_M-1)}{\pi}|\operatorname{Im}(z)|\frac{|1-z^2|^{l_M-2}}{(1-|z|^2)^{l_M+1}}\int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^{\infty}(1-u^2)^{\frac{l_M-3}{2}}du\times\\ \left[I_{|z|^2}(L,l_M+1)-I_{|z|^2}(M-1,l_M+1)\right].\end{aligned} \tag{4.2.171}$$

The mean density of real eigenvalues for the real induced Jacobi ensemble can be written as:

$$\begin{aligned}\rho^{\mathbb{R}}_{\mathrm{IndSpherical},N}(x) = & \frac{1}{B\left(\frac{l_M}{2},\frac{1}{2}\right)}\frac{I_{x^2}(L,l_M)-I_{x^2}(M-1,l_M)}{(1-x^2)} \\ & + \frac{1}{B\left(\frac{l_M}{2},\frac{M}{2}\right)}|x|^{M-1}(1-x^2)^{\frac{l_M-2}{2}}I_{x^2}\Big(\frac{M-1}{2},\frac{l_M}{2}\Big) \\ & + \frac{1}{B\left(\frac{l_M}{2},\frac{L+1}{2}\right)}x^L(1-x^2)^{\frac{l_M-2}{2}}I_{1-x^2}\Big(\frac{l_M}{2},\frac{L}{2}\Big). \end{aligned} \tag{4.2.172}$$

We can distinguish four different asymptotic regimes depending on the rectangularity parameter L and the parameter controlling the size of the truncations l_M. Figure 4.4 shows the eigenvalue distribution of the real induced Jacobi ensemble in the four asymptotic regimes.

Strong rectangularity and strong non-orthogonality

In the regime of strong rectangularity and strong non-orthogonality the size of the rectangularity parameter L grows proportionally with matrix size: $L = N\alpha$, while the size of the induced matrix grows proportionally with the size of the orthogonal matrix: $K = kN$. This implies the following: $l_N = (k-1)N$, $M = (\alpha+1)N$ and $l_M = (k-\alpha-1)N$. In addition set $\mu_1 := \frac{\alpha}{k-1}$ and $\mu_2 := \frac{\alpha+1}{k}$. In the truncated orthogonal matrix ,used to generate the induced Jacobi ensemble, the number of deleted rows and columns grows proportionally with matrix size.

In the limit of large matrix dimensions the mean density of complex eigenvalues is supported on an annulus of width $\sqrt{\mu_2}-\sqrt{\mu_1}$. Thus the density has two cut-offs, the inner edge with radius $r^{\mathrm{in}} = \sqrt{\mu_1}$ and the outer edge with radius $r^{\mathrm{out}} = \sqrt{\mu_2}$, while the mean density of real eigenvalues is supported on two disjoint intervals: $[-\sqrt{\mu_2}, -\sqrt{\mu_1}\,]$ and $[\sqrt{\mu_1}, \sqrt{\mu_2}\,]$ and the average number of real eigenvalues is to leading order given by: $\sqrt{\frac{l_M}{2\pi}}\Big(\ln\Big(\frac{\sqrt{K}+\sqrt{M}}{\sqrt{K}-\sqrt{M}}\Big) - \ln\Big(\frac{\sqrt{l_N}+\sqrt{L}}{\sqrt{l_N}-\sqrt{L}}\Big)\Big)$. Thus the average number of real eigenvalues is of order $\sqrt{N}$. Close to the edges of the eigenvalue support the mean eigenvalue densities exhibit universal behavior. Similarly closing down on the real line $z = x+iy$ with $y = \frac{u}{\sqrt{n+L}}$ the mean density of complex eigenvalues becomes:

$$\lim_{N\to\infty}\frac{1}{l_M}\,\rho^{\mathbb{C}}_{\mathrm{IndJacobi}}(z) = \big(\rho^{\mathbb{R}}_{\mathrm{IndJacobi}}(x)\big)^2 h\big(u\rho^{\mathbb{R}}_{\mathrm{IndJacobi}}(x)\big), \tag{4.2.173}$$

where

$$h(u) = 4\pi|u|\,\mathrm{e}^{4\pi u^2}\,\mathrm{erfc}(\sqrt{4\pi}|u|) \tag{4.2.174}$$

$$\rho^{\mathbb{R}}_{\mathrm{IndJacobi}}(x) = \lim_{N\to\infty}\frac{1}{\sqrt{l_M}}\rho^{\mathbb{R}}_{\mathrm{IndJacobi},N}(z). \tag{4.2.175}$$

Furthermore in the limit of large matrix dimensions after unfolding the respective correlation kernels one recovers the same limiting expressions as in the case of the real Ginibre ensemble. A detailed account of the limiting expression for the correlation kernel in the bulk, can be found in section B.3, see theorem B.3.1. Moreover,

Theorem 4.2.18. *In the regime of strong rectangularity and strong non-orthogonality, $L = N\alpha$ and $K = kN$, in the limit of large matrix dimension, the mean density of complex eigenvalues is given by:*

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{l_M}\,\rho^{\mathbb{C}}_{IndJacobi,N}(z)\\ =&\frac{1}{\pi}\frac{1}{(1-|z|^2)^2}\,\left[\Theta\big(|z|-\sqrt{\mu_1}\,\big)-\Theta\big(|z|-\sqrt{\mu_2}\,\big)\right] =: \rho^{\mathbb{C}}_{IndJacobi}(z),\end{aligned} \tag{4.2.176}$$

while the mean density of real eigenvalues is given by:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{\sqrt{l_M}}\,\rho^{\mathbb{R}}_{IndJacobi,N}(z)\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{1-x^2}\,\left[\Theta\big(|\,x|-\sqrt{\mu_1}\,\big)-\Theta\big(|\,x|-\sqrt{\mu_2}\,\big)\right] =: \rho^{\mathbb{R}}_{IndJacobi}(z).\end{aligned} \tag{4.2.177}$$

At the edges of $z^{in} = \big(\sqrt{\mu_1}-\frac{\xi}{\sqrt{l_M}}\big)\,\mathrm{e}^{i\phi}$ and $z^{out} = \big(\sqrt{\mu_2}+\frac{\xi}{\sqrt{l_M}}\big)\,\mathrm{e}^{i\phi}$ of the complex eigenvalue density:

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{l_M}\rho^{\mathbb{C}}_{IndJacobi,N}\big(z^{in}\,\big) =&\pi\rho^{\mathbb{C}}_{IndJacobi}\big(\sqrt{\mu_1}\big)\frac{1}{2\pi}\,\mathrm{erfc}\left(\sqrt{2\pi\rho^{\mathbb{C}}_{IndJacobi}\big(\sqrt{\mu_1}\big)}\xi\right)\\ =&\frac{1}{2\pi}\frac{1}{(1-\mu_1)^2}\,\mathrm{erfc}\left(\frac{\sqrt{2}}{\sqrt{1-\mu_1}}\xi\right)\end{aligned} \tag{4.2.178}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{l_M}\rho^{\mathbb{C}}_{IndJacobi,N}\big(z^{out}\,\big) =&\pi\rho^{\mathbb{C}}_{IndJacobi}\big(\sqrt{\mu_2}\big)\frac{1}{2\pi}\,\mathrm{erfc}\left(\sqrt{2\pi\rho^{\mathbb{C}}_{IndJacobi}\big(\sqrt{\mu_2}\big)}\xi\right)\\ =&\frac{1}{2\pi}\frac{1}{(1-\mu_2)^2}\,\mathrm{erfc}\left(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\right).\end{aligned} \tag{4.2.179}$$

At the edges $x^{in} = \sqrt{\mu_1}-\frac{\xi}{\sqrt{n+L}}$ and $x^{out} = \sqrt{\mu_2}+\frac{\xi}{\sqrt{n+L}}$ of the real eigenvalue *density:*

$$\lim_{N\to\infty}\frac{1}{\sqrt{l_M}}\rho^{\mathbb{R}}_{IndJacobi,N}(x^{in}) \tag{4.2.180}$$

$$=\frac{1}{\sqrt{2\pi}}\Big[\frac{1}{1-\mu_1}\operatorname{erfc}\big(\frac{\sqrt{2}}{\sqrt{1-\mu_1}}\xi\big)+\frac{1}{2\sqrt{2}}\frac{1}{1-\mu_1}e^{\frac{u^2}{1-\mu_1}}\operatorname{erfc}\Big(-\frac{1}{\sqrt{1-\mu_1}}\xi\Big)\Big]$$

$$\lim_{N\to\infty}\frac{1}{\sqrt{l_M}}\rho^{\mathbb{R}}_{IndJacobi,N}(x^{out}) \tag{4.2.181}$$

$$=\frac{1}{2\pi}\Big[\frac{1}{1-\mu_2}\operatorname{erfc}\big(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\big)+\frac{1}{2\sqrt{2}}\frac{1}{1-\mu_2}e^{\frac{u^2}{1-\mu_2}}\operatorname{erfc}\Big(-\frac{1}{\sqrt{1-\mu_2}}\xi\Big)\Big].$$

Closing down on the real line with scaling $z=x+iy$ and $y=\frac{u}{\sqrt{l_M}}$ the complex eigenvalue density becomes:

$$\lim_{N\to\infty}\frac{1}{l_M}\rho^{\mathbb{C}}_{IndJacobi,N}\big(x+i\frac{u}{\sqrt{l_M}}\big)=\sqrt{\frac{2}{\pi}}|u|\frac{1}{(1-x^2)^3}\,\mathrm{e}^{\frac{2u^2}{1-x^2}}\operatorname{erfc}\big(\sqrt{2}\frac{|u|}{1-x^2}\big). \tag{4.2.182}$$

Proof. First we analyze the asymptotic behavior in the bulk away from the real line. From theorem A.2.1 it follows that:

$$\lim_{N\to\infty}I_{|z|^2}(L,l_M+1)=\Theta\Big(|z|-\sqrt{\mu_1}\Big) \tag{4.2.183}$$

$$\lim_{N\to\infty}I_{|z|^2}(M-1,l_M+1)=\Theta\Big(|z|-\sqrt{\mu_2}\Big) \tag{4.2.184}$$

Note with $z=x+iy$ and $a=1+\frac{4y^2}{(1-x^2-y^2)^2}$:

$$\begin{aligned}
&2|y|\frac{|1-z^2|^{l_M-2}}{(1-|z|^2)^{l_M+1}}\int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^1(1-u^2)^{\frac{l_M-3}{2}}du\\
&=\frac{1}{2}\frac{\big((1-x^2-y^2)^2+4y^2\big)^{\frac{l_M}{2}}}{(1-x^2-y^2)^{l_M+1}}\frac{2|y|}{|1-z^2|}\int_{\frac{4y^2}{(1-x^2-y^2)^2+4y^2)}}^1u^{-\frac{1}{2}}(1-u)^{\frac{l_M-3}{2}}du\\
&=\frac{1}{2}\frac{1}{(1-|z|^2)^2}\Big(1+\frac{4y^2}{(1-x^2-y^2)^2}\Big)^{\frac{l_M-1}{2}}\Big(\frac{4y^2}{(1-x^2-y^2)^2+4y^2}\Big)^{\frac{1}{2}}\\
&\int_0^{\frac{(1-x^2-y^2)^2}{(1-x^2-y^2)^2+4y^2}}u^{-\frac{1}{2}}(1-u)^{\frac{l_M-3}{2}}du\\
&=\frac{1}{2}\frac{1}{(1-|z|^2)^2}\,a^{\frac{l_M-1}{2}}(1-a^{-1})^{\frac{1}{2}}\int_0^{a^{-1}}u^{-\frac{1}{2}}(1-u)^{\frac{l_M-3}{2}}du.
\end{aligned} \tag{4.2.185}$$

Now change variables $w = \log(\frac{a^{-1}}{u})$, then:

$$
\begin{aligned}
&2|y|\frac{|1-z^2|^{l_M-2}}{(1-|z|^2)^{l_M+1}}\int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^{\infty}(1-u^2)^{\frac{l_M-3}{2}}du\\
=&\frac{1}{2}\frac{1}{(1-|z|^2)^2}(1-a^{-1})^{\frac{1}{2}}\int_0^{\infty}\mathrm{e}^{-\frac{l_M-3}{2}}(1-a^{-1}\,\mathrm{e}^{-w})^{-\frac{1}{2}}du\\
\sim&\frac{1}{(1-|z|^2)^2}\frac{1}{l_M-1},
\end{aligned}
\tag{4.2.186}
$$

where we applied Watson's lemma. Thus we derived equation (4.2.176). All other quantities follow due to similar derivations as for the real induced spherical ensemble in the regime of strong rectangularity with strong spherical component.

□

Strong rectangularity and partially weak non-orthogonality

In the regime of strong rectangularity and partially weak non-orthogonality the rectangularity parameter is scaled as before: $L = N\alpha$, while the parameter controlling the orthogonality of the induced Jacobi matrix l_M is kept fixed. Note that strong rectangularity implies: $l_N = K - N = K - \frac{1}{1+\alpha}(K - l_M) = \frac{\alpha}{1+\alpha}K - \frac{l_M}{1+\alpha}$. The number of deleted rows l_M is kept fixed, while strong rectangularity implies that the number of deleted columns grows proportionally with matrix size l_N.

In this regime the eigenvalues of $A_{\text{IndJacobi}}$ lie close to the unit circle. The average number of real eigenvalues is to leading order given by:

$$
\mathcal{N}^{R}_{\text{IndJacobi}} = \frac{1}{B\big(\frac{l_M}{2},\frac{1}{2}\big)}\Big(\ln\Big(\frac{\sqrt{K}+\sqrt{M}}{\sqrt{K}-\sqrt{M}}\Big) - \ln\Big(\frac{\sqrt{l_N}+\sqrt{L}}{\sqrt{l_N}-\sqrt{L}}\Big)\Big). \tag{4.2.187}
$$

Thus the average number of real eigenvalues is of order $\log(N)$. Furthermore a new type of correlation kernel emerges, extending the number of known universality classes for non-hermitian random matrix ensembles for $\beta = 1$. The explicit form of this correlation kernel is omitted, due to space restrictions. More precisely,

Theorem 4.2.19. *Set $z = (1-\frac{y}{N}),\mathrm{e}^{i\phi}$, then in the regime of strong rectangularity and partially weak non-orthogonality, $L = N\alpha$ and $l_M = O(1)$, in the limit of large matrix dimensions, the mean density of complex eigenvalues is given by:*

$$
\lim_{N\to\infty}\frac{1}{N^2}\rho^{\mathbb{C}}_{IndJacobi,N}\big((1-\frac{y}{N})\,\mathrm{e}^{i\phi}\big) = \frac{1}{\pi}\frac{\Gamma(2y(\alpha+1),l_M+1)-\Gamma(2y\alpha,l_M+1)}{\Gamma(l_M)}. \tag{4.2.188}
$$

In addition set $x = 1 - \frac{u}{N}$, then in the limit of large matrix dimensions the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty} \frac{1}{N}\rho^{\mathbb{R}}_{IndJacobi,N}\big(1 - \frac{u}{N}\big) = \frac{1}{2uB(\frac{L}{2},\frac{1}{2})}\frac{\Gamma(2y(\alpha+1), l_M+1) - \Gamma(2y\alpha, l_M+1)}{\Gamma(l_M)}$$
$$+\frac{1}{2u}\big[u(\alpha+1)\big]^{\frac{l_M}{2}}\,\mathrm{e}^{-u(\alpha+1)}\,\frac{1}{\Gamma\big(\frac{l_M}{2}\big)}\frac{\Gamma\big(2u(\alpha+1),\frac{l_M}{2}\big)}{\Gamma\big(\frac{l_M}{2}\big)}$$
$$+\frac{1}{2u}\big[u\alpha\big]^{\frac{l_M}{2}}\,\mathrm{e}^{-u\alpha}\,\frac{1}{\Gamma\big(\frac{l_M}{2}\big)}\frac{\Gamma\big(2u\alpha,\frac{l_M}{2}\big)}{\Gamma\big(\frac{l_M}{2}\big)} := \rho^{\mathbb{R}}_{IndJacobi}(u) \tag{4.2.189}$$

This implies for small $u << 1$:

$$\rho^{\mathbb{R}}_{IndJacobi}(u) \sim \frac{1}{2u}\frac{\big[u(\alpha+1)\big]^{\frac{l_M}{2}}}{\Gamma\big(\frac{l_M}{2}\big)} + \frac{1}{2u}\frac{\big[u\alpha\big]^{\frac{l_M}{2}}}{\Gamma\big(\frac{l_M}{2}\big)}. \tag{4.2.190}$$

Proof. The limiting expression for the complex eigenvalue density follows from the equivalent theorem for the complex induced Jacobi ensemble and proof of theorem 4.2.18. Applying theorem A.2.2 to the four incomplete beta functions, as well as rewriting the beta function gives the limiting expression for the real eigenvalue density. □

Almost square matrices and strong non-orthogonality

In the asymptotic regime of almost square matrices with strong non-orthogonality the rectangularity parameter is kept fixed $L = O(1)$, while $K = kN$ grows proportionally with matrix size. This implies the relations $l_M = (k-1)N - L$ as well as $l_N = (k-1)N$. Set $\mu_2 = \frac{1}{k}$. The number of deleted rows and columns is proportional to matrix size N.

In the limit of large matrix dimensions the complex eigenvalues are distributed across a disk around the origin with radius $r^{\text{out}} = \sqrt{\mu_2}$, while the density of real eigenvalues is supported on the interval $[-\sqrt{\mu_2}, \sqrt{\mu_2}]$. The average number of real eigenvalues is to leading order given by:

$$\mathcal{N}^R_{\text{IndJacobi}} = \sqrt{\frac{l_M}{2\pi}}\ln\Big(\frac{\sqrt{K}+\sqrt{M}}{\sqrt{K}-\sqrt{M}}\Big). \tag{4.2.191}$$

and thus is of order $\sqrt{N}$. Close to the edges of the eigenvalue support the mean eigenvalue densities exhibit universal behavior. Similarly closing down on the real line $z = x + iy$ with $y = \frac{u}{\sqrt{n+L}}$ the mean density of complex eigenvalues becomes:

$$\lim_{N\to\infty} \frac{1}{l_M}\, \rho^{\mathbb{C}}_{\text{IndJacobi}}(z) = \big(\rho^{\mathbb{R}}_{\text{IndJacobi}}(x)\big)^2 h\big(u\rho^{\mathbb{R}}_{\text{IndJacobi}}(x)\big), \tag{4.2.192}$$

where

$$h(u) = 4\pi|u|\,\mathrm{e}^{4\pi u^2}\,\mathrm{erfc}(\sqrt{4\pi}|u|) \tag{4.2.193}$$

$$\rho^{\mathbb{R}}_{\text{IndJacobi}}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{l_M}} \rho^{\mathbb{R}}_{\text{IndJacobi},N}(z). \tag{4.2.194}$$

In the limit of large matrix dimensions the correlation kernels exhibit universal behavior after unfolding. Furthermore at the origin after unfolding the correlation kernels of the real induced Ginibre ensemble can be recovered. For more details see section B.3, theorem B.3.2. Moreover,

Theorem 4.2.20. *In the regime of almost square matrices and strong non-orthogonality, $L = O(1)$ and $K = kN$ in the limit of large matrix dimension the mean density of complex eigenvalues is given by:*

$$\lim_{N\to\infty} \frac{1}{l_M}\, \rho^{\mathbb{C}}_{\mathit{IndJacobi},N}(z) = \frac{1}{\pi}\frac{1}{(1-|z|^2)^2}\,\Theta\big(\sqrt{\mu_2} - |z|\,\big) =: \rho^{\mathbb{C}}_{\mathit{IndJacobi}}(z), \tag{4.2.195}$$

while the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty} \frac{1}{\sqrt{l_M}}\, \rho^{\mathbb{R}}_{\mathit{IndJacobi},N}(z) = \frac{1}{\sqrt{2\pi}}\frac{1}{1-x^2}\,\Theta\big(\sqrt{\mu_2} - |x|\,\big) =: \rho^{\mathbb{R}}_{\mathit{IndJacobi}}(z). \tag{4.2.196}$$

At the outer circular edge $z^{out} = \big(\sqrt{\mu_2} + \frac{\xi}{\sqrt{l_M}}\big)\,\mathrm{e}^{i\phi}$ of the complex eigenvalue density:

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{l_M}\, \rho^{\mathbb{C}}_{\mathit{IndJacobi},N}\big(z^{out}\,\big) =& \pi\rho^{\mathbb{C}}_{\mathit{IndJacobi}}\big(\sqrt{\mu_2}\,\big)\frac{1}{2\pi}\,\mathrm{erfc}\,\Big(\sqrt{2\pi\rho^{\mathbb{C}}_{\mathit{IndJacobi}}\big(\sqrt{\mu_2}\big)}\xi\Big)\\ =&\frac{1}{2\pi}\frac{1}{(1-\mu_2)^2}\,\mathrm{erfc}\,\Big(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\Big)\end{aligned} \tag{4.2.197}$$

At the outer edge $x^{out} = \sqrt{\mu_2} + \frac{\xi}{\sqrt{n+L}}$ of the real eigenvalue density:

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{\sqrt{l_M}} \rho^{\mathbb{R}}_{\mathit{IndJacobi},N}\big(x^{out}\big) \qquad (4.2.198)\\ =&\frac{1}{2\pi}\Big[\frac{1}{1-\mu_2}\,\mathrm{erfc}\,\big(\frac{\sqrt{2}}{\sqrt{1-\mu_2}}\xi\big) + \frac{1}{2\sqrt{2}}\frac{1}{1-\mu_2}\,\mathrm{e}^{\frac{u^2}{1-\mu_2}}\,\mathrm{erfc}\,\Big(-\frac{1}{\sqrt{1-\mu_2}}\xi\Big)\Big]\end{aligned}$$

Closing down on the real line with scaling $z = x + iy$ and $y = \frac{u}{\sqrt{l_M}}$ the complex *eigenvalue density becomes:*

$$\lim_{N\to\infty} \frac{1}{l_M} \rho^{\mathbb{C}}_{IndJacobi,N}\big(x + i\frac{u}{\sqrt{l_M}}\big) = \sqrt{\frac{2}{\pi}}|u| \frac{1}{(1-x^2)^3}\,\mathrm{e}^{\frac{2u^2}{1-x^2}}\,\mathrm{erfc}\,\big(\sqrt{2}\frac{|u|}{1-x^2}\big). \tag{4.2.199}$$

Almost square matrices and weak non-orthogonality

Finally the rectangularity parameter $L = O(1)$ as well as the parameter controlling the orthogonality $l_M = O(1)$ are kept fixed. This implies that the number of deleted rows and columns is kept fixed. In the regime of almost square matrices and weak non-orthogonality the eigenvalues are again distributed in the vicinity of the unit circle. In the limit of large matrix dimensions the mean eigenvalue densities of truncations of random orthogonal matrices in the regime of weak non-orthogonality are recovered [KSŻ10]. The average number of real eigenvalues is to leading order given by:

$$\mathcal{N}^R_{\mathrm{IndJacobi}} \sim \frac{1}{B\big(\frac{l_M}{2},\frac{1}{2}\big)} \ln\Big(\frac{\sqrt{K}+\sqrt{M}}{\sqrt{K}-\sqrt{M}}\Big). \tag{4.2.200}$$

Equally the limiting eigenvalue correlation kernels coincide with the limiting correlation kernels found in [KSŻ10] for truncations of orthogonal matrices in the regime of weak non-orthogonality. These coincide with the correlation kernel of the complex induced Jacobi ensemble from theorem 3.2.18.

Theorem 4.2.21. *Set $z = (1-\frac{y}{N})\,\mathrm{e}^{i\phi}$, then in the regime of almost square and weak non-orthogonality in the limit of large matrix dimensions the mean density of complex eigenvalues is given by:*

$$\lim_{N\to\infty} \frac{1}{N^2} \rho^{\mathbb{C}}_{IndJacobi,N}\big((1-\frac{y}{N})\,\mathrm{e}^{i\phi}\big) = \frac{1}{\pi}\frac{\gamma(2y, l_M+1)}{\Gamma(l_M)}. \tag{4.2.201}$$

In addition set $x = 1-\frac{u}{N}$, then in the limit of large matrix dimensions the mean density of real eigenvalues is given by:

$$\lim_{N\to\infty} \frac{1}{N} \rho^{\mathbb{R}}_{IndJacobi,N}\big(1-\frac{u}{N}\big) \tag{4.2.202}$$
$$= \frac{1}{2uB(\frac{L}{2},\frac{1}{2})}\frac{\gamma(2u,l_M+1)}{\Gamma(l_M)} + \frac{1}{2u}u^{\frac{l_M}{2}}\,\mathrm{e}^{-u}\,\frac{1}{\Gamma\big(\frac{l_M}{2}\big)}\frac{\gamma\big(2u,\frac{l_M}{2}\big)}{\Gamma\big(\frac{l_M}{2}\big)} \quad := \rho^{\mathbb{R}}_{IndJacobi}(u)$$

This implies for small $u << 1$:

$$\rho^{\mathbb{R}}_{IndJacobi}(u) \sim \frac{1}{2u}\frac{u^{\frac{l_M-2}{2}}}{\Gamma\big(\frac{l_M}{2}\big)}, \tag{4.2.203}$$

while for large $u >> 0$:

$$\rho^{\mathbb{R}}_{IndJacobi}(u) \sim \frac{1}{2u}\frac{1}{B\left(\frac{l_M}{2},\frac{1}{2}\right)}. \tag{4.2.204}$$

4.3 Summary of results

4.3.1 The real induced spherical ensemble

- The eigenvalue jpdf weight function of a real induced spherical matrix:

$$w_{\text{IndSpherical},1}(z) = \frac{z^L}{|1+z^2|^{\frac{n+L+1}{2}}}\left(\int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty}(1+u^2)^{-\frac{n+L+2}{2}}du\right)^{\frac{1}{2}}. \tag{4.3.1}$$

- The finite N mean density of complex eigenvalues for the real induced spherical ensemble:

$$\rho^{\mathbb{C}}_{\text{IndSpherical},N}(z) = \frac{(n+L)(n+L+1)}{\pi}|\operatorname{Im}(z)|\frac{|1+|z|^2|^{n+L-2}}{|1+z^2|^{n+L+1}}\times \tag{4.3.2}$$
$$\int_{\frac{2|\operatorname{Im}(z)|}{|1+z^2|}}^{\infty}(1+u^2)^{-\frac{n+L+1}{2}}du\left[I_{\frac{|z|^2}{1+|z|^2}}(L,n-1)-I_{\frac{|z|^2}{1+|z|^2}}(M-1,n-N)\right].$$

- The finite N mean density of real eigenvalues for the real induced spherical ensemble:

$$\begin{aligned}\rho^{\mathbb{R}}_{\text{IndSpherical},N}(x) = &\frac{1}{B\left(\frac{n+L}{2},\frac{1}{2}\right)}\frac{1}{1+x^2}\left[I_{\frac{x^2}{1+x^2}}(L,n)-I_{\frac{x^2}{1+x^2}}(M,n-N+1)\right]\\ &+\frac{1}{B\left(\frac{n-N+1}{2},\frac{M}{2}\right)}\frac{|x|^{M-1}}{(1+x^2)^{\frac{n+L+1}{2}}}I_{\frac{x^2}{1+x^2}}\left(\frac{M+1}{2},\frac{n-N+2}{2}\right)\\ &+\frac{1}{B\left(\frac{n}{2},\frac{L+1}{2}\right)}\frac{x^L}{(1+x^2)^{\frac{n+L+1}{2}}}I_{\frac{1}{1+x^2}}\left(\frac{n+1}{2},\frac{L}{2}\right).\end{aligned} \tag{4.3.3}$$

- Eigenvalue support in the four distinct asymptotic regimes:

  Strong rectangularity, strong spherical component: Annulus
  Strong rectangularity, weak spherical component: complex plane without disk around origin
  Almost square, strong spherical component: disk around origin
  Almost square, weak spherical component: whole complex plane

- Limiting correlation kernel in the bulk in the two regimes of strong rectangularity: real Ginibre after unfolding, see theorem B.2.1 and theorem

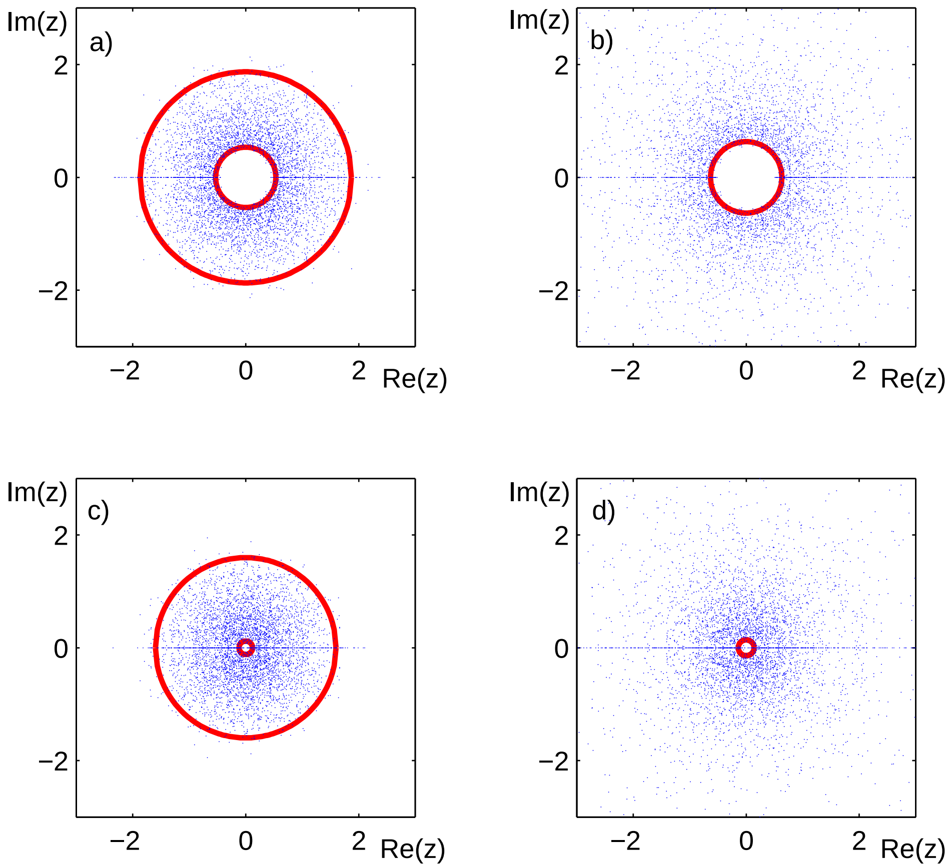


Figure 4.2: Spectra of matrices pertaining to the induced spherical ensemble of real matrices for dimension $N = 100$ and a) $L = 40$, $n - N = 40$, b) $L = 40$, $n - N = 0$, c) $L = 2$, $n - N = 40$, d) $L = 2$, $n - N = 2$. Each plot consists of data from 50 independent realizations. The circles of radius $r_{\text{in}} = \sqrt{L/n}$ (inner one) and $r_{\text{out}} = \sqrt{M/(N-n)}$ (outer one) are depicted to guide the eye.

B.2.2.

- Limiting correlation kernel in the bulk in the two regimes of almost square matrices: real Ginibre after unfolding, see theorem B.2.3 and theorem B.2.4. Limiting correlation kernel at the origin in the two regimes of almost square matrices: real induced Ginibre at origin.

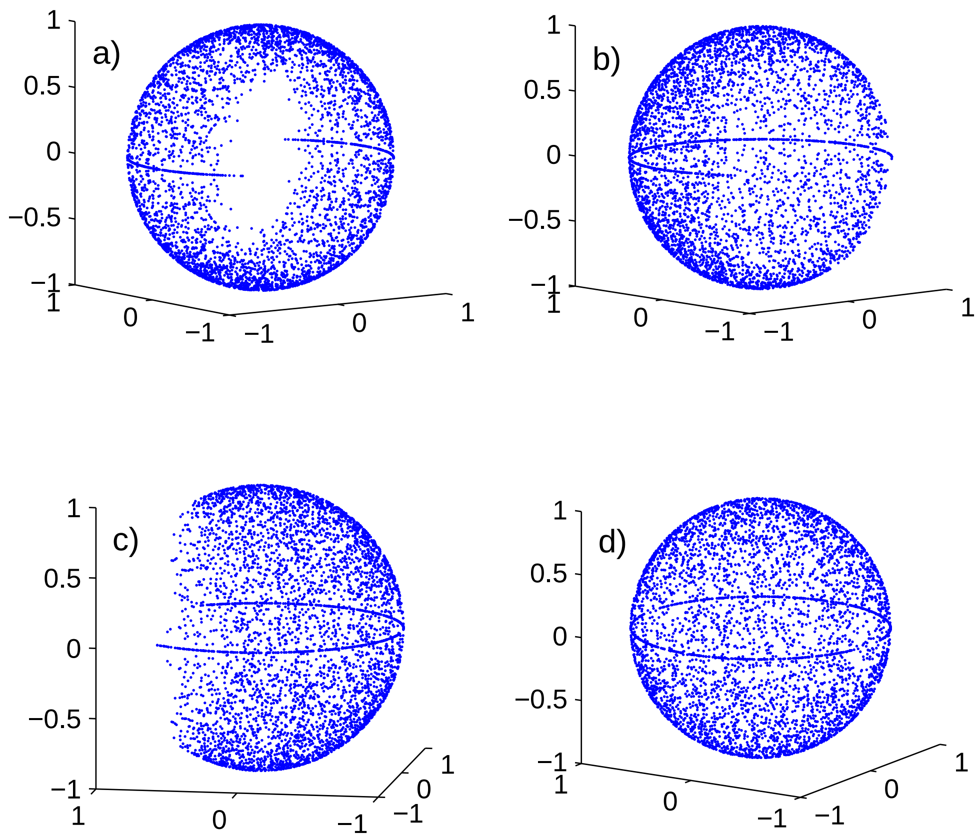


Figure 4.3: Spectra of matrices pertaining to the induced spherical ensemble of real matrices for dimension $N = 100$ and a) $L = 40$, $n - N = 40$, b) $L = 40$, $n - N = 0$, c) $L = 2$, $n - N = 40$, d) $L = 2$, $n - N = 2$ after inverse stereographical projection to the sphere.

4.3.2 The real induced Jacobi ensemble

- The eigenvalue jpdf weight function of a real induced Jacobi matrix:

$$w_{\mathrm{IndJacobi},1}(z) = z^L |1-z^2|^{\frac{l_M-2}{2}} \Big(\int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^{1} (1-u^2)^{\frac{l_M-3}{2}} du \Big)^{\frac{1}{2}}. \tag{4.3.4}$$

- The finite N mean density of complex eigenvalues for the real induced Jacobi ensemble:

$$\rho^{\mathbb{C}}_{\mathrm{IndJacobi},N}(z) = \frac{2l_M(l_M-1)}{\pi} |\operatorname{Im}(z)| \frac{|1-z^2|^{l_M-2}}{(1-|z|^2)^{l_M+1}} \int_{\frac{2|\operatorname{Im}(z)|}{|1-z^2|}}^{\infty} (1-u^2)^{\frac{l_M-3}{2}} du \times$$
$$\left[I_{|z|^2}(L, l_M+1) - I_{|z|^2}(M-1, l_M+1) \right]. \tag{4.3.5}$$

- The finite N mean density of real eigenvalues for the real induced Jacobi ensemble:

$$\begin{aligned}\rho^{\mathbb{R}}_{\mathrm{IndSpherical},N}(x) = & \frac{1}{B\big(\frac{l_M}{2},\frac{1}{2}\big)} \frac{I_{x^2}(L,l_M) - I_{x^2}(M-1,l_M)}{(1-x^2)} \\ & + \frac{1}{B\big(\frac{l_M}{2},\frac{M}{2}\big)} |x|^{M-1}(1-x^2)^{\frac{l_M-2}{2}} I_{x^2}\Big(\frac{M-1}{2},\frac{l_M}{2}\Big) \\ & + \frac{1}{B\big(\frac{l_M}{2},\frac{L+1}{2}\big)} x^L (1-x^2)^{\frac{l_M-2}{2}} I_{1-x^2}\Big(\frac{l_M}{2},\frac{L}{2}\Big). \end{aligned} \tag{4.3.6}$$

- Limiting mean eigenvalue densities in the regime of strong rectangularity:

$$\lim_{N\to\infty} \rho^{\mathbb{R}}_{\mathrm{IndGin},N}(\sqrt{N}x) = \frac{1}{\sqrt{2\pi}} \left[\Theta(|x|-\sqrt{\alpha}) - \Theta(|x|-\sqrt{\alpha+1}) \right] \tag{4.3.7}$$

$$\lim_{N\to\infty} \rho^{\mathbb{C}}_{\mathrm{IndGin},N}(\sqrt{N}z) = \frac{1}{\pi} \left[\Theta(|z|-\sqrt{\alpha}) - \Theta(|z|-\sqrt{\alpha+1}) \right]. \tag{4.3.8}$$

- Eigenvalue support in the four distinct asymptotic regimes:

  Strong rectangularity, strong non-orthogonality: Annulus
  Strong rectangularity, partially weak non-orthogonality: eigenvalues distributed close to the unit circle
  Almost square, strong non-orthogonality: disk around origin with radius less than one
  Almost square, weak non-orthogonality: eigenvalues distributed close to the unit circle

- Limiting correlation kernel in the bulk in the regime of strong rectangularity and strong non-orthogonality: real Ginibre after unfolding, see theorem B.3.1.

- Limiting correlation kernel in the bulk in the regime of strong rectangularity and partially weak non-orthogonality: new correlation kernel, see theorem 3.2.16.

- Limiting correlation kernel in the bulk in the regime of almost square matrices and strong non-orthogonality: real Ginibre after unfolding, see theorem B.3.2.

  Limiting correlation kernel at the origin in the regime of almost square matrices and strong non-orthogonality: real induced Ginibre at the origin, see theorem B.3.2.

- Limiting correlation kernel in the bulk in the regime of almost square matrices and weak non-orthogonality: truncations of random orthogonal matrices [KSŻ10], see theorem 3.2.18.

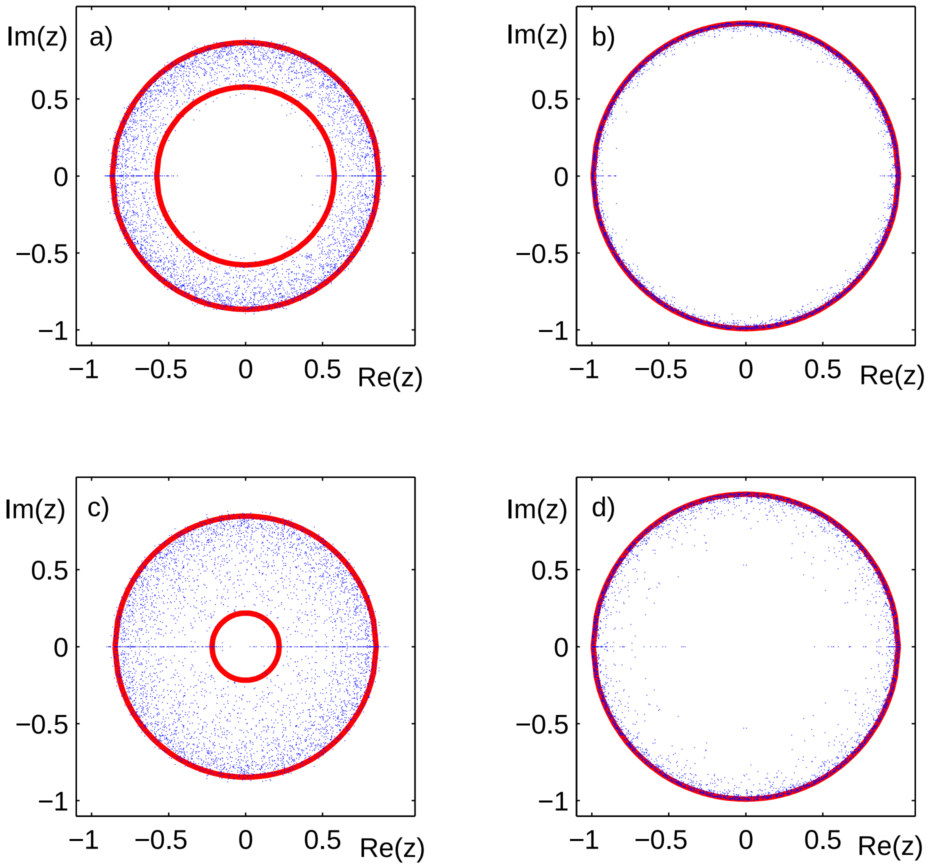


Figure 4.4: Spectra of matrices pertaining to the induced Jacobi ensemble of real matrices for dimension $N = 100$ and a) $L = 20$, $l_M = 40$, b) $L = 20$, $l_M = 2$, c) $L = 2$, $l_M = 20$, d) $L = 2$, $l_M = 2$. Each plot consists of data from 50 independent realizations. The circles of radius $r_{\rm in} = \sqrt{L/K}$ (inner one) and $r_{\rm out} = \sqrt{M/K}$ (outer one) are depicted to guide the eye.

Chapter 5

Conclusions

Even though a number of specific examples of real asymmetric random matrix ensemble are completely solved (Ginibre [Gin65], Truncations [SŻ00, KSŻ10], Spherical [FK09, FM11], Chiral [Ake11]) the theory of non-hermitian random matrices is still far from being as thoroughly understood as its Hermitian counterpart.

In this work have introduced a family of three non-hermitian random matrix ensembles: the induced Ginibre ensemble, the induced Jacobi ensemble and the induced spherical ensemble through an inducing procedure. The inducing procedure consisted of quadratizing rectangular random matrices. As a result we obtained infinitely many generalizations of the three main solvable non-hermitian random matrix ensembles (besides the chiral models).

Furthermore each induced random matrix ensemble was solved for $\beta = 1, 2$, meaning that the eigenvalue jpdf, the correlation functions and eigenvalue densities were derived. In the case of complex matrix entries the correlation functions could be expressed in closed form using determinants. By construction all three examples of the family of induced random matrix ensemble are rotationally invariant, which in the case of the complex induced ensembles made the application of the method of orthogonal polynomials particularly straightforward. In the case of real induced ensembles the method of skew-orthogonal polynomials was used to express the correlation functions as Pfaffians. In addition an extensive asymptotic analysis was undertaken for each real and complex ensemble.

For the induced complex and real Ginibre ensemble two asymptotic regimes were studied: strong rectangularity and almost square matrices. In the regime of strong rectangularity the eigenvalues were uniformly distributed on an annulus around the origin and universal behavior of the eigenvalue statistics was established, meaning that the limiting correlation kernels of the induced Ginibre ensemble coincided with the limiting correlation kernels of the Ginibre ensemble. Similarly in the regime of almost square matrices a new correlation kernel was found at the origin. At a distance of $\sqrt{N}$ away from the origin universality was again established for the induced Ginibre ensemble in the regime of almost square matrices.

As a contrast for the induced real and complex spherical ensemble four distinct asymptotic regimes were identified. The main distinction between these four regimes was the support of the eigenvalue density, which was Cauchy distributed on either a ring around the origin, a disk around the origin, the whole complex plane or the complex plane except a disk around the origin. Again in the strongly rectangular regimes we established universality for the eigenvalue statistics of interest, while in the regime of almost square matrices at the origin the correlation kernels of the induced Ginibre ensemble in the regime of almost square matrices were recovered.

The most fascinating asymptotic behavior was found for the induced real and complex Jacobi ensemble. Again four distinct asymptotic regimes were identified, depending on the rectangularity parameter as well as the number of deleted rows and columns in the random Haar distributed matrix, used to generate the induced Jacobi ensemble. Eigenvalues were to leading order either distributed on an annulus, a disk around the origin or close to the unit circle. While in the regime of strong orthogonality (strong unitarity) we established universality of the eigenvalue statistics, recovering the induced Ginibre universality class after appropriate scaling, in the regime of almost square matrices with weak non-orthogonality (non-unitarity) the eigenvalue statistics of truncations of random orthogonal (unitary matrices) were found. One of the main results of this work is the discovery of a new universality class in the regime of weak and partially weak non-orthogonality (non-unitarity). In particular a new limiting correlation kernel was discovered.

In tune with the title of this work each ensemble boasted an asymptotic regime, in which the respective eigenvalues densities were supported on a ring around the origin. This form of the spectrum suggests a comparison with the model (1.2.3) of Feinberg–Zee, for which the 'single ring' theorem was proven [FZ97, GZ11].

Our model formally thus belongs to the Feinberg–Zee class, with the potential:

$$V(G^\dagger G) = G^\dagger G - L \log G^\dagger G, \tag{5.0.1}$$

but due to the log function the assumption that the potential is polynomial is not satisfied.

Thus does the induced family of real and complex ensembles not only provide specific examples of the single ring distribution for the general version of the Feinberg–Zee model. In addition the methods used in [FZ97, GZ11] only give access to the leading order behavior of the mean eigenvalue density. Finer points of the eigenvalue statistics like higher order correlation functions, as well as the distribution and average number of real eigenvalues are beyond its reach.

Finally the discovery of a new class of correlation kernel in the induced Jacobi ensemble serves to highlight the fact, that universality in the context of non-hermitian random matrices still remains an open and extremely challenging problem and a classification of all universality classes is still out of reach.

Appendix A

Asymptotics of special functions

In the following the asymptotic expansions are carried out using either the Laplace method in the case of real integrals or the saddle-point method in the case of complex integrals. Those methods are generally employed to compute asymptotic expansions of integrals of the form:

$$I(N) = \int_a^b \exp(-Np(t))q(t)dt, \tag{A.0.1}$$

for large N. The main idea behind the Laplace method is that the principal contribution to the integral will come from the peak value of $\exp(-Np(t))$ which will occur at the minimum value t_0 of $p(t)$. For large N the peak is going to be very sharp and one can replace the function $p(t)$ and $q(t)$ by the leading terms of their Taylor expansion around t_0 and only take into account the integral on a neighborhood of t_0.

A.1 Gamma function asymptotics

Most results in this section are already known, but will be rederived in a unified manner, using Laplace and saddle-point methods.

Theorem A.1.1. *Let $x \in (0, \infty)$ as well as $a \in \mathbb{R}_{>-0.5}$, then:*

$$\lim_{N\to\infty} \frac{1}{\Gamma(Na)}\gamma(Nx, Na) = \Theta(x-a). \tag{A.1.1}$$

Proof. As our aim is to apply the Laplace method, we need to rewrite the lower incomplete gamma function as follows:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) = 1 - \frac{1}{\Gamma(Na)}\int_{Nx}^{\infty} t^{Na-1}\,\mathrm{e}^{-t}\,dt \tag{A.1.2}$$

A change variables $t = s + Nx$ furthermore gives:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) = 1 - \frac{\mathrm{e}^{-Nx}}{\Gamma(Na)} \int_0^\infty (s + Nx)^{Na-1}\,\mathrm{e}^{-s}\,ds. \tag{A.1.3}$$

Additionally we set $s = Nu$, which leads to:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) = 1 - \frac{N^{Na}\,\mathrm{e}^{-Nx}}{\Gamma(Na)} \int_0^\infty \mathrm{e}^{-N\left[u - a\ln(u+x)\right]}\,\frac{1}{u+x}\,du. \tag{A.1.4}$$

The integral is now in the right form for applying the Laplace method. Thus set:

$$\begin{aligned} I(N) &:= \int_0^\infty \mathrm{e}^{-Np(u)}\,q(u)du \\ p(u) &:= u - a\ln(u+x) \\ q(u) &:= \frac{1}{u+x}. \end{aligned} \tag{A.1.5}$$

It is necessary to find the absolut minimum of $p(u)$. Note that:

$$p'(u) = 1 - \frac{a}{u+x} \tag{A.1.6}$$

and $p'(a-x) = 0$. As a result have to distinguish between three different cases:
1. $0 < x < a$
$u_0 = a - x$ is the absolute minimum of the function $p(u)$. We can expand around $a - x$: as $u \to u_0$ from the right:

$$p(u) - p(a-x) \sim \frac{1}{2a}(u - (a-x))^2 \qquad \text{and} \qquad q(u) \sim \frac{1}{a}. \tag{A.1.7}$$

This gives us then:

$$\begin{aligned} I(N) &\sim \int_{a-x-\epsilon}^{a-x+\epsilon} \mathrm{e}^{-N\left[a-x-a\ln a+\frac{1}{2a}(u-(a-x))^2\right]}\,a^{-1}du \quad \text{for} \quad N\to\infty \\ &= a^{Na-1}\,\mathrm{e}^{-N(a-x)} \int_{a-x-\epsilon}^{a-x+\epsilon} \mathrm{e}^{-\frac{N}{2a}\left(u-(a-z^2)\right)^2}\,du. \end{aligned} \tag{A.1.8}$$

A change of variable $t = \sqrt{\frac{N}{a}}(u - (a-x))$ leads to:

$$\begin{aligned} I(N) &\sim \frac{1}{a}a^{Na}\,\mathrm{e}^{-N(a-x)}\sqrt{\frac{a}{N}} \int_{-\sqrt{\frac{N}{a}}\epsilon}^{\sqrt{\frac{N}{a}}\epsilon} \mathrm{e}^{-\frac{1}{2}t^2}\,dt \\ &\sim \frac{a^{Na}}{\sqrt{aN}}\,\mathrm{e}^{-N(a-x)} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}t^2}\,dt = \frac{a^{Na}}{\sqrt{aN}}\sqrt{2\pi}\,\mathrm{e}^{-N(a-x)}\,. \end{aligned} \tag{A.1.9}$$

Additionally using Stirling's formula:

$$\Gamma(Na) \sim \mathrm{e}^{-Na}(Na)^{Na}\sqrt{\frac{2\pi}{Na}}. \tag{A.1.10}$$

Finally:

$$1 - \frac{1}{\Gamma(Na)}\gamma(Nx, Na) \sim \frac{\mathrm{e}^{-Na})(Na)^{Na}}{\mathrm{e}^{-Na}(Na)^{Na}}\sqrt{\frac{Na}{2\pi}}\sqrt{\frac{2\pi}{Na}} = 1 \quad \text{for} \quad N \to \infty. \tag{A.1.11}$$

2. $x = a$

$u_0 = 0 = a - x$ is the local and absolute minimum. In addition u_0 is one of the limits of integration. Consequently the peak of the function lies on the edge of integration and we have to halve our approximation to obtain the correct result. Thus starting from:

$$I(N) \sim \int_0^{\epsilon} \mathrm{e}^{aN\ln a - \frac{N}{2a}u^2}\, a^{-1}du. \tag{A.1.12}$$

Again we perform a change of variable: $u = \sqrt{\frac{a}{N}}t$ and obtain:

$$I(N) \sim a^{Na-1}\sqrt{\frac{a}{N}}\int_0^{\frac{N}{a}\epsilon} \mathrm{e}^{\frac{1}{2}t^2}\, dt \sim \frac{a^{Na}}{\sqrt{aN}}\int_0^{\infty} \mathrm{e}^{\frac{1}{2}t^2}\, dt = \frac{1}{2}\frac{a^{Na}}{\sqrt{aN}}\sqrt{2\pi}. \tag{A.1.13}$$

All in all:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) \sim 1 - \frac{\mathrm{e}^{-Na}(Na)^{Na}}{\Gamma(Na)}\frac{\sqrt{2\pi}}{2\sqrt{aN}}. \tag{A.1.14}$$

As before we use the approximation for large values of the Gamma-function and obtain:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) \sim \frac{1}{2} \quad \text{for} \quad N \to \infty. \tag{A.1.15}$$

3. $x > a$

In the interval $[0, \infty)$, $u_0 = 0$ is the absolute minimum of $p(u)$, but it is not a local minimum as $p'(0) \neq 0$. Hence:

$$p(u) \sim p(0) + p'(0)u = -a\ln x + \left(1 - \frac{a}{x}\right)u. \tag{A.1.16}$$

which results in:

$$\begin{aligned} I(N) &\sim \int_0^{\epsilon} \mathrm{e}^{-N\left[-a\ln x + (1-\frac{a}{x})u\right]}\,\frac{1}{x}\, du = (x)^{Na-1}\int_0^{\epsilon} \mathrm{e}^{-N(1-\frac{a}{x})u}\, du \\ &\sim x^{Na-1}\frac{1}{N(1-\frac{a}{x})}\int_0^{\infty} \mathrm{e}^{-t}\, dt = \frac{x^{Na-1}}{N(1-\frac{a}{x})}. \end{aligned} \tag{A.1.17}$$

Consequently using Stirling's formula again yields:

$$\frac{1}{\Gamma(Na)}\gamma(Nx, Na) \sim 1 \quad \text{for} \quad N \to \infty. \tag{A.1.18}$$

□

Theorem A.1.2. *Let $a \in (0, \infty)$ as well as $\xi \in \mathbb{R}$. In addition set $x_{out} = \sqrt{a} + \frac{\xi}{\sqrt{N}}$ as well as $x_{in} = \sqrt{a} - \frac{\xi}{\sqrt{N}}$, then:*

$$\lim_{N\to\infty} \frac{1}{\Gamma(Na)}\gamma(Nx_{out}^2, Na) = 1 - \frac{1}{2}\operatorname{erfc}\left(\sqrt{2}\xi\right) \tag{A.1.19}$$

$$\lim_{N\to\infty} \frac{1}{\Gamma(Na)}\gamma(Nx_{in}^2, Na) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{2}\xi\right). \tag{A.1.20}$$

Proof. Again we start by rewriting the lower incomplete gamma function:

$$\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}^2, Na) = 1 - \frac{1}{\Gamma(Na)}\int_{Nx_{\text{out}}^2}^{\infty} t^{Na-1}\,\mathrm{e}^{-t}\,dt. \tag{A.1.21}$$

A change of variable $t = Ns$ leads to:

$$\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}, Na) = 1 - \frac{N^a}{\Gamma(Na)}\int_{a+2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}}^{\infty} s^{Na-1}\,\mathrm{e}^{-Ns}\,ds. \tag{A.1.22}$$

Another change of variable $u = s - 2\sqrt{\frac{a}{N}}\xi - \frac{\xi^2}{N}$ results in:

$$\begin{aligned}&\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}, Na) \\ =&1 - \frac{N^{Na}}{\Gamma(Na)}\int_a^{\infty} \mathrm{e}^{-N\left\{u+2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}-a\ln\left(u+2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}\right)\right\}}\,\frac{1}{u+2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}}du.\end{aligned} \tag{A.1.23}$$

Now we can apply the Laplace method with $w = u + 2\sqrt{\frac{a}{N}}\xi + \frac{\xi^2}{N}$:

$$\begin{aligned}p(w) &= w - a\ln w \qquad \text{and} \qquad q(w) = \frac{1}{w}\\ p'(w) &= 1 - \frac{a}{w}\\ p''(w) &= \frac{a}{w^2}.\end{aligned}$$

The minimum value of $p(w)$ in the interval $[a, \infty)$ is a with $p'(a) = 0$. Hence:

$$p(w) \sim a + a\ln(a) + \frac{1}{2a}\left(w - a\right)^2 \quad \text{and} \quad q(w) \sim \frac{1}{a} \tag{A.1.24}$$

This results in:

$$\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}, Na) \sim 1 - \frac{[Na]^{Na}e^{-Na}}{a\Gamma(Na)} \int_a^{a+\epsilon} e^{-\frac{N}{2a}\left(u+2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}-a\right)^2} du. \quad (A.1.25)$$

Substituting $b = \sqrt{\frac{a}{N}}\big(u - a\big)$ then gives:

$$\begin{aligned}
\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}, Na) &\sim 1 - \frac{[Na]^{Na}e^{-Na}}{a[Na]^{Na}e^{-Na}}\sqrt{\frac{Na}{2\pi}}\sqrt{\frac{a}{N}}\int_{2\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}b^2}\, db \\
&\sim 1 - \frac{1}{\sqrt{2\pi}}\int_{2\xi}^{\infty} \mathrm{e}^{-\frac{1}{2}b^2}\, db = 1 - \frac{1}{\sqrt{\pi}}\int_{\sqrt{2}\xi}^{\infty} \mathrm{e}^{-b^2}\, db \\
&= 1 - \frac{1}{2}\,\mathrm{erfc}(\sqrt{2}\xi). \qquad (A.1.26)
\end{aligned}$$

All in all we derived:

$$\lim_{N\to\infty} \frac{1}{\Gamma(Na)}\gamma(Nx_{\text{out}}, Na) = \frac{1}{2}\,\mathrm{erfc}(\sqrt{2}\xi). \quad (A.1.27)$$

For the second limit start from:

$$\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{in}}^2, Na) = 1 - \frac{N^{Na}}{\Gamma(Na)}\int_{a-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}}^{\infty} s^{Na-1}\,\mathrm{e}^{-Ns}\, ds. \quad (A.1.28)$$

Furthermore rewriting this expression gives:

$$\begin{aligned}
&\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{in}}^2, Na) \qquad (A.1.29)\\
=&1 - \frac{N^{Na}}{\Gamma(Na)}\int_a^{\infty} \mathrm{e}^{-N\left\{u-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}-a\ln\left(u-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}\right)\right\}}\,\frac{1}{u-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}}du.
\end{aligned}$$

Another application of the Laplace method yields:

$$\begin{aligned}
\frac{1}{\Gamma(Na)}\gamma(Nx_{\text{in}}^2, Na) &\sim 1 - \frac{N^{Na}}{a\Gamma(Na)}\int_a^{a+\epsilon} e^{-N\left[a-a\ln(a)+\frac{1}{2a}\left(u-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}-a\right)^2\right]} du \\
&= 1 - \frac{[Na]^{Na}e^{-Na}}{a\Gamma(Na)}\int_0^{\epsilon} e^{-\frac{N}{2a}\left(u-2\sqrt{\frac{a}{N}}\xi+\frac{\xi^2}{N}\right)^2} du \\
&\sim 1 - \frac{1}{\sqrt{2\pi}}\int_{-2\xi}^{\infty} e^{-\frac{1}{2}s^2} ds \\
&= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\sqrt{2}\xi\right).
\end{aligned}$$

As a consequence:

$$\lim_{N\to\infty} \frac{1}{\Gamma(Na)}\gamma(Nx_{\text{in}}, Na) = \frac{1}{2}\,\mathrm{erfc}(\sqrt{2}\xi). \quad (A.1.30)$$

$\square$

Theorem A.1.3. *Let u, z_k, z_l be complex numbers with $\sqrt{a} \leq |u| \leq \sqrt{a+1}$, let $a \in (0,\infty)$ and set $\lambda_k = \sqrt{N}u + z_k$ and $\lambda_l = \sqrt{N}u + z_l$, then:*

$$\lim_{N\to\infty}\left[\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l, Na) - \frac{1}{\Gamma(N(a+1))}\gamma\big(\lambda_k\bar{\lambda}_l, N(a+1)\big)\right] = 1. \tag{A.1.31}$$

Proof. It is sufficient to derive the asymptotic behavior of:

$$\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l, Na) = 1 - \frac{1}{\Gamma(Na)}\int_{N|u|^2+\sqrt{N}(u\bar{z}_l+\bar{u}z_j)+z_k\bar{z}_l}^{\infty} t^{Na-1}\,\mathrm{e}^{-t}\,dt. \tag{A.1.32}$$

Rewriting the above expression yields:

$$\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l, Na) = 1 - \frac{N^{Na}}{\Gamma(Na)}\int_0^{\infty} e^{-N[t+w-a\ln(t+w)]}\frac{1}{t+w}dt \tag{A.1.33}$$

With $w := |u|^2 + \frac{u\bar{z}_l+\bar{u}z_j}{\sqrt{N}} + \frac{z_k\bar{z}_l}{N}$ as well as:

$$p(t) = t + w - a\ln(t+w) \qquad \text{and} \qquad q(t) = \frac{1}{t+w}, \tag{A.1.34}$$

the integral is in the required form for the saddle point method. We note that $p(s)$ is analytic for $\mathrm{ph}(t+w) \in (-\pi, \pi]$ on the sliced complex plane and $q(s)$ is analytic in $C\backslash\{-w\}$.

$$p'(t) = 1 - \frac{a}{t+w} \tag{A.1.35}$$

Hence the saddle point lies at $t_0 = a - w$. We again have to consider three cases:
1. $0 < |u| < \sqrt{a}$
The saddle point lies in the interior of the curve. $\mathrm{Re}\{p(t)\}$ attains its minimum for $s_0 = a - w$. We can now deform the path C according to Cauchy's theorem such that it passes through the saddle point t_0. The path of steepest descent occurs when $\mathrm{Im}\{p(t)\}$ is constant. Our new path goes parallel to the real line passing through t_0:

$$p(t) \sim p(t_0) + \frac{1}{2a}\big(t - t_0\big)^2 \qquad \text{and} \qquad q(t) \sim q(t_0).$$

We choose a path that goes parallel through the real line and passes through t_0.

All in all we obtain:

$$
\begin{aligned}
1-\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l,Na) &\sim \frac{N^{Na}}{a\Gamma(Na)}\int_{t_0+C_1}^{t_0+C_2} \mathrm{e}^{-N\left[a-a\ln a+\frac{1}{2a}\left(t-(a-w)\right)^2\right]}\,dt \\
&= \frac{(Na)^{Na}\,\mathrm{e}^{-Na}}{a\Gamma(Na)}\sqrt{\frac{a}{N}}\int_{\sqrt{\frac{N}{a}}C_1}^{\sqrt{\frac{N}{a}}C_2} e^{\frac{1}{2}s^2}ds \\
&\sim \frac{(N\alpha)^{N\alpha}\exp(-N\alpha)}{\alpha\Gamma(N\alpha)}\sqrt{\frac{2\pi\alpha}{N}} \sim 1
\end{aligned} \tag{A.1.36}
$$

2. $|u|=\sqrt{\alpha}$
The saddle point $s_0=0$ now lies on the limit of integration. As a result the contour of the integral does not need to be deformed. We obtain:

$$
\begin{aligned}
\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l,Na) &\sim 1-\frac{N^{Na}}{a\Gamma(Na)}\int_0^{C_2} e^{-N\left[a-a\ln(a)+\frac{1}{2a}\left(t-(a-w)\right)^2\right]}dt \\
&\sim 1-\frac{(Na)^{Na}\,\mathrm{e}^{-N\alpha}}{a\Gamma(Na)}\frac{1}{2}\sqrt{\frac{2\pi}{N}} \sim \frac{1}{2}.
\end{aligned} \tag{A.1.37}
$$

3. $|u|>\sqrt{\alpha}$
$\mathrm{Re}\{p(t)\}$ attains its minimum value for $t_0=0$, but $p'(t_0)=1-\frac{a}{w}\neq 0$. This leads to:

$$
p(t)\sim p(0)+(1-\frac{a}{w})t \qquad \text{and} \qquad q(t)\sim q(0)=\frac{1}{w}, \tag{A.1.38}
$$

which implies:

$$
\begin{aligned}
\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l,Na) &\sim 1-\frac{N^{Na}}{\Gamma(Na)}\int_0^{C_2} e^{-N\left[w-a\ln w+(1-\frac{a}{w})t\right]}\frac{1}{w}dt \\
&\sim 1-\frac{\sqrt{2\pi}}{w\sqrt{N}(1-\frac{a}{w})}
\end{aligned} \tag{A.1.39}
$$

□

Theorem A.1.4. *Let u,z_k,z_l be complex numbers with $|u|=1$, let $a\in(0,\infty)$ and set $\lambda_k=\sqrt{N(a+1)}u+z_k$ and $\lambda_l=\sqrt{N(a+1)}u+z_l$, then:*

$$
\lim_{N\to\infty}\left[\frac{1}{\Gamma(Na)}\gamma(\lambda_k\bar{\lambda}_l,Na)-\frac{1}{\Gamma(N(a+1))}\gamma\big(\lambda_k\bar{\lambda}_l,N(a+1)\big)\right]=\mathrm{erfc}\left(\frac{z_j\bar{u}+\bar{z}_k u}{\sqrt{2}}\right).
$$

Setting $\lambda_k=\sqrt{Na}u-z_k$ and $\lambda_l=\sqrt{Na}u-z_l$, yields the same limiting behavior.

Proof. Again we start by rewriting the lower incomplete gamma function. Together with the proof of theorem A.1.3 it is sufficient to derive the asymptotic behavior of:

$$
\begin{aligned}
&\frac{1}{\Gamma(N(a+1))}\gamma(\lambda_k\bar{\lambda}_l, N(a+1)) \\
&= 1 - \frac{1}{\Gamma(N(a+1))}\int_{N(a+1)+\sqrt{N(a+1)}(u\bar{z}_l+\bar{u}z_k)+z_k\bar{z}_l}^{\infty} t^{N(a+1)-1}\,\mathrm{e}^{-t}\,dt \\
&= 1 - \frac{N^{N(a+1)}}{\Gamma(N(a+1))}\int_{a+1}^{\infty} \mathrm{e}^{-N\left[s+\sqrt{\frac{a+1}{N}}(u\bar{z}_l+\bar{u}z_k)+\frac{z_k\bar{z}_l}{N}-(a+1)\ln\left(s+\sqrt{\frac{a+1}{N}}(u\bar{z}_l+\bar{u}z_k)+\frac{z_k\bar{z}_l}{N}\right)\right]} \\
&\times \frac{1}{s+\sqrt{\frac{a+1}{N}}(u\bar{z}_l+\bar{u}z_k)+\frac{z_k\bar{z}_l}{N}}ds. \qquad \text{(A.1.40)}
\end{aligned}
$$

Applying the saddle-point method then yields:

$$
\begin{aligned}
&\frac{1}{\Gamma(N(a+1))}\gamma(\lambda_k\bar{\lambda}_l, N(a+1)) \\
\sim &1 - \frac{[(\alpha+1)N]^{N(a+1)}\,\mathrm{e}^{-N(\alpha+1)}}{(a+1)\Gamma(N(a+1))}\int_{a+1}^{a+1+\epsilon} \mathrm{e}^{-\frac{N}{2(a+1)}\left(s+\sqrt{\frac{a+1}{N}}(u\bar{z}_l+\bar{u}z_k)+\frac{z_k\bar{z}_l}{N}-(a+1)\right)^2} ds \\
&\sim 1 - \frac{1}{\sqrt{2\pi}}\int_{u\bar{z}_l+\bar{u}z_k}^{\infty} e^{\frac{1}{2}-t^2}dt = 1 - \frac{1}{2}\,\mathrm{erfc}\,\Big(\frac{u\bar{z}_l+\bar{u}z_k}{2}\Big). \qquad \text{(A.1.41)}
\end{aligned}
$$

□

A.2 Beta function asymptotics

Theorem A.2.1. *Let x be real with $0 < x < 1$ and let $a, b \in \mathbb{R}_{>0.5}$, then:*

$$
\lim_{a,b\to\infty} I_x(a,b) = \Theta\Big(\frac{\alpha}{1+\alpha} - x\Big), \quad \textit{where} \quad \lim_{a,b\to\infty}\frac{a}{b} = \alpha. \qquad \text{(A.2.1)}
$$

Proof. The incomplete beta function is defined as:

$$
I_x(a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt. \qquad \text{(A.2.2)}
$$

Using Stirling's approximation formula yields:

$$
\begin{aligned}
B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sim \frac{e^{-a}a^a\sqrt{\frac{2\pi}{a}}e^{-b}b^b\sqrt{\frac{2\pi}{b}}}{e^{-a-b}(a+b)^{a+b}\sqrt{\frac{2\pi}{a+b}}} \\
&= \sqrt{\frac{2\pi}{b}}\sqrt{\frac{a+b}{a}}\frac{b^b a^a}{(a+b)^{a+b}} = \sqrt{\frac{2\pi}{a}}\sqrt{1+\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^a\Big(\frac{1}{1+\alpha}\Big)^b. \qquad \text{(A.2.3)}
\end{aligned}
$$

Furthermore:

$$\int_0^x t^{a-1}(1-t)^{b-1}dt = \int_0^x e^{a\left(\ln(t)+\frac{1}{\alpha}\ln(1-t)\right)}\frac{1}{t(1-t)}dt. \tag{A.2.4}$$

Now set:

$$p(t) = \ln(t) + \frac{1}{\alpha}\ln(1-t)\,, \qquad q(t) = \frac{1}{t(1-t)}. \tag{A.2.5}$$

Comparison with the integral from equation (A.2.3) shows, that the function $p(t)$ takes its minimum at $t_0 = \frac{\alpha}{1+\alpha}$. Thus it is necessary to distinguish three cases:
1. $x > \frac{\alpha}{1+\alpha}$
Expanding around the minimum t_0 gives:

$$p(t) \sim \ln(\alpha) - \frac{1+\alpha}{\alpha}\ln(1+\alpha) + \frac{(1+\alpha)^3}{\alpha^2}\big(t - \frac{\alpha}{1+\alpha}\big)^2, \quad q(t) \sim \frac{(1+\alpha)^2}{\alpha}\,,$$

which in turn yields:

$$I_x(a,b) \sim \sqrt{\frac{a}{2\pi}}\frac{1}{\sqrt{1+\alpha}}\frac{(1+\alpha)^2}{\alpha}\int_{\frac{\alpha}{1+\alpha}-\epsilon}^{\frac{\alpha}{1+\alpha}+\epsilon} \mathrm{e}^{-a\frac{(1+\alpha)^3}{\alpha^2}\left(t-\frac{\alpha}{1+\alpha}\right)^2}\,dt \sim 1 \tag{A.2.6}$$

2. $x = \frac{\alpha}{1+\alpha}$
Similarly using the expansions:

$$p(t) \sim \ln(\alpha) - \frac{1+\alpha}{\alpha}\ln(1+\alpha) + \frac{(1+\alpha)^3}{\alpha^2}\big(t - \frac{\alpha}{1+\alpha}\big)^2, \quad q(t) \sim \frac{(1+\alpha)^2}{\alpha}\,,$$

yields:

$$I_x(a,b) \sim \sqrt{\frac{a}{2\pi}}\frac{1}{\sqrt{1+\alpha}}\frac{(1+\alpha)^2}{\alpha}\int_{\frac{\alpha}{1+\alpha}-\epsilon}^{\frac{\alpha}{1+\alpha}} \mathrm{e}^{-a\frac{(1+\alpha)^3}{\alpha^2}\left(t-\frac{\alpha}{1+\alpha}\right)^2}\,dt \sim \frac{1}{2}. \tag{A.2.7}$$

3. $x < \frac{\alpha}{1+\alpha}$
Again we need to expand:

$$p(t) \sim \ln(\alpha) - \frac{1+\alpha}{\alpha}\ln(1+\alpha) + \big(\frac{1}{x} - \frac{1}{\alpha(1-x)}\big)(t-x)\,, \quad q(t) \sim \frac{(1+\alpha)^2}{\alpha}\,,$$

, then:

$$I_x(a,b) \sim \sqrt{\frac{a}{2\pi}}\frac{1}{\sqrt{1+\alpha}}\frac{(1+\alpha)^2}{\alpha}\int_{x-\epsilon}^{x+\epsilon} e^{a\left(\frac{1}{x}-\frac{1}{\alpha(1-x)}\right)(t-x)}dt \sim 0.$$

$\square$

Theorem A.2.2. *Let x be real with $-\infty < x < \infty$, $a, b \in \mathbb{R}_{>0.5}$ and set $y = (1 - \frac{x}{N})\,\mathrm{e}^{i\phi}$, then:*

$$\lim_{N\to\infty} I_{|y|^2}(N\tilde{a}, b) = \frac{\Gamma(2x\tilde{a}, b)}{\Gamma(b)} \quad \textit{for} \quad b \quad \textit{fixed.} \tag{A.2.8}$$

Proof. It is convenient to study the related integral:

$$1 - I_{|y|^2}(N\tilde{a}, b) = \frac{1}{B(N\tilde{a}, b)} \int_{1-\frac{2x}{N}+\frac{x^2}{N^2}}^{1} t^{N\tilde{a}-1}(1-t)^{b-1}dt. \tag{A.2.9}$$

The change of variables $t = 1 - \left(\frac{2x}{N} - \frac{x^2}{N^2}\right)s$ yields:

$$1 - I_{|y|^2}(N\tilde{a}, b) \sim \frac{(2x)^b}{N^b B(N\tilde{a}, b)} \int_0^1 s^{b-1}\left(1 - \frac{2x}{N}\right)^{N\tilde{a}-1} ds. \tag{A.2.10}$$

In addition note:

$$\begin{aligned}(N\tilde{a})^b B(N\tilde{a}, b) = (N\tilde{a})^b \frac{\Gamma(N\tilde{a})\Gamma(b)}{\Gamma(N\tilde{a}+b)} &\sim (N\tilde{a})^b \frac{\Gamma(b)\,\mathrm{e}^{-N\tilde{a}}(N\tilde{a})^{N\tilde{a}}\sqrt{\frac{2\pi}{N\tilde{a}}}}{\mathrm{e}^{-N\tilde{a}+b}(N\tilde{a}+b)^{N\tilde{a}+b}\sqrt{\frac{2\pi}{N\tilde{a}}}} \\ &\sim \Gamma(b)\,\mathrm{e}^b\left(1 + \frac{b}{N\tilde{a}}\right)^{-N\tilde{a}-b+\frac{1}{2}} \sim \Gamma(b).\end{aligned} \tag{A.2.11}$$

Hence:

$$1 - I_{|y|^2}(N\tilde{a}, b) \sim \frac{(2x)^b}{\Gamma(b)} \int_0^1 s^{b-1} e^{-2x\tilde{a}} ds. \tag{A.2.12}$$

□

Theorem A.2.3. *Let $z \in \mathbb{C}$ with $0 < |z| < 1$, $a, b \in \mathbb{R}_{>0.5}$, then:*

$$\lim_{a\to\infty,\ b\ \textit{fixed}} I_z(a, b) = 0 \qquad \textit{and} \qquad \lim_{b\to\infty,\ a\ \textit{fixed}} I_z(a, b) = 1. \tag{A.2.13}$$

Proof. We start with the first limit. From equation (A.2.11) we obtain:

$$B(a, b) \sim \Gamma(b) a^{-b} \quad \text{for} \quad a \to \infty\text{ , b fixed.} \tag{A.2.14}$$

Furthermore we change variables $w = \log(\frac{x}{t})$, such that:

$$I_z(a, b) = \frac{z^a}{B(a, b)} \int_0^\infty e^{-aw}\left(1 - ze^{-w}\right)^{b-1} dw \tag{A.2.15}$$

is in the right form to apply Watson's lemma with:

$$h(t) = \left(1 - ze^{-w}\right)^{b-1} \sim (1 - z)^{b-1} t^{\frac{1-1}{1}}. \tag{A.2.16}$$

Thus implying:

$$\int_0^\infty e^{-aw}\left(1 - ze^{-w}\right)^{b-1} dw \sim \frac{(1-z)^{b-1}}{a}, \tag{A.2.17}$$

which yields:

$$I_z(a,b) \sim \Gamma(b) z^a (1-z)^{b-1} a^b. \tag{A.2.18}$$

The second relation follows from:

$$I_z(a,b) = 1 - I_{1-z}(b,a). \tag{A.2.19}$$

$\square$

Theorem A.2.4. *Let $a, b \in \mathbb{R}_{>0.5}$ with $a, b \to \infty$ and $\frac{a}{b} = \alpha = O(1)$. In addition set $\mu = \frac{\alpha}{\alpha+1}$ and $z = \sqrt{\mu} + \frac{\xi}{\sqrt{a}}$. Then:*

$$\lim_{N\to\infty} I_{|z|^2}(a,b) = 1 - \frac{1}{2}\operatorname{erfc}\Big(\frac{\sqrt{2}}{\sqrt{\mu(1-\mu)}}\xi\Big). \tag{A.2.20}$$

Proof. Using the definition of the incomplete beta function:

$$\begin{aligned} I_{|z|^2}(a,b) = &\frac{1}{B(a,b)} \int_0^{\mu + 2\xi\sqrt{\frac{\mu}{a}} + \frac{\xi^2}{a}} t^{a-1}(1-t)^{b-1} dt \\ =& I_\mu(a,b) + \frac{1}{B(a,b)} \int_\mu^{\mu + 2\xi\sqrt{\frac{\mu}{a}} + \frac{\xi^2}{a}} t^{a-1}(1-t)^{b-1} dt \\ =& I_\mu(a,b) + \frac{1}{B(a,b)} \int_0^{2\xi\sqrt{\frac{\mu}{a}} + \frac{\xi^2}{a}} (\mu+t)^{a-1}(1+\mu-t)^{b-1} dt \end{aligned} \tag{A.2.21}$$

Using theorem A.2.1 and its proof:

$$\begin{aligned} &I_{|z|^2}(a,b) \\ \sim&\frac{1}{2} + \sqrt{\frac{a}{2\pi}}\sqrt{1-\mu}\mu^{-a}(1-\mu)^{-b} \int_0^{2\xi\sqrt{\frac{\mu}{a}} + \frac{\xi^2}{a}} (\mu+t)^{a-1}(1-\mu-t)^{b-1} dt \\ \sim&\frac{1}{2} + \sqrt{\frac{a}{2\pi}}\sqrt{1-\mu}\mu^{-1}(1-\mu)^{-1} \int_0^{2\xi\sqrt{\frac{\mu}{a}} + \frac{\xi^2}{a}} \Big(1+\frac{t}{\mu}\Big)^{a-1}\Big(1-\frac{t}{1-\mu}\Big)^{b-1} dt. \end{aligned} \tag{A.2.22}$$

Changing variables $t = \frac{1}{\sqrt{a}}(1-\mu)s$ then yields:

$$\begin{aligned} & I_{|z|^2}(a,b) && \text{(A.2.23)} \\ \sim & \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sqrt{1-\mu}\mu^{-1}\int_0^{2\xi\sqrt{\mu}(1-\mu)} \left(1+\frac{t}{\sqrt{a}}\frac{1-\mu}{\mu}\right)^{a-1}\left(1-\frac{t}{\sqrt{a}}\right)^{b-1} dt \\ \sim & \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sqrt{1-\mu}\mu^{-1}\int_0^{2\xi\sqrt{\mu}(1-\mu)} \mathrm{e}^{(a-1)\ln\left(1+\frac{t}{\sqrt{a}}\frac{1-\mu}{\mu}\right)}\,\mathrm{e}^{(b-1)\ln\left(1-\frac{t}{\sqrt{a}}\right)}\,dt \end{aligned}$$

Asymptotic expansion of the logarithmic terms gives:

$$I_{|z|^2}(a,b) \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-\mu}}{\mu}\int_0^{2\xi\sqrt{\mu}(1-\mu)} \mathrm{e}^{-\frac{1}{2}s^2\frac{\sqrt{1-\mu}}{\mu}}\,ds. \tag{A.2.24}$$

□

Corollary A.2.5. *Let $a, b \in \mathbb{R}_{>0.5}$ with $Na, Nb \to \infty$ and $\frac{a}{b} = \alpha = O(1)$. In addition set $z = \sqrt{\alpha} + \frac{\xi}{\sqrt{a}}$. Then:*

$$\lim_{N\to\infty} I_{\frac{|z|^2}{1+|z|^2}}(Na, Nb) = \lim_{N\to\infty} J_{|z|^2}(Na, Nb) = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{\sqrt{2}}{\sqrt{\alpha(1+\alpha)}}\xi\right), \tag{A.2.25}$$

where

$$J_z(a,b) = \frac{1}{B(a,b)}\int_0^z \frac{t^{a-1}}{(1+t)^{a+b}}dt. \tag{A.2.26}$$

Proof. In the following it is easier to work with:

$$J_{|z|^2}(Na, Na+Nb) = \frac{1}{B(Na,Nb)}\int_0^{|z|^2} \frac{t^{Na-1}}{(1+t)^{N(a+b)}}dt \tag{A.2.27}$$

instead of the definition of the incomplete beta function. The first step is expanding:

$$\frac{t^{a-1}}{(1+t)^{a+b}} = \mathrm{e}^{N\left(a\log(t)-(a+b)\log(1+t)\right)}\,t^{-1} \tag{A.2.28}$$

around its the minimum $t_0 = \alpha$ as well as using the beta function asymptotics.

Consequently:

$$
\begin{aligned}
&J_{|z|^2}(Na, Na+Nb)\\
\sim&\frac{1}{2}+\sqrt{\frac{a}{2\pi}}\frac{1}{\sqrt{1+\mu_1}}\Big(\frac{1+\alpha}{\alpha}\Big)^{Na}(1+\alpha)^{Nb}\int_0^{2\sqrt{\frac{\alpha}{a}}\xi}\frac{(\alpha+t)^{Na-1}}{(1+\alpha+t)^{N(a+b)}}dt\\
\sim&\frac{1}{2}+\sqrt{\frac{a}{2\pi}}\frac{1}{\alpha\sqrt{1+\alpha}}\int_0^{2\sqrt{\frac{\alpha}{a}}\xi}\frac{(1+\frac{t}{\alpha})^{Na-1}}{(1+\frac{t}{1+\alpha})^{N(a+b)}}dt\\
\sim&\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\alpha(1+\alpha)}}\int_0^{2\xi}e^{-\frac{1}{2}s^2\frac{1}{\alpha(1+\alpha)}}ds.
\end{aligned}
\tag{A.2.29}
$$

□

Theorem A.2.6. *Set* $z_k = u + \frac{s_k}{\sqrt{l_M}}$, $k = 1, \dots, N$ *where* $u, s_k \in \mathbb{C}$ *with* $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$ *and* $\mu_1 = \frac{\alpha_1}{\alpha_1+1}$, $\alpha_1 = \frac{L}{l_M}$, $\mu_2 = \frac{\alpha_2}{\alpha_2+1}$, $\alpha_2 = \frac{M}{l_M}$. *Then*

$$\lim_{N\to\infty}\Big[I_{z_k\bar{z}_l}\big(L, l_M+1\big) - I_{z_k\bar{z}_l}\big(M, l_M+1\big)\Big] = 1. \tag{A.2.30}$$

Theorem A.2.7. *Set* $z_k = u + \frac{s_k}{\sqrt{n+L}}$, $k = 1, \dots, N$ *where* $u, s_k \in \mathbb{C}$ *with* $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$ *and* $\mu_1 = \frac{L}{n}$, $\mu_2 = \frac{\alpha_2}{\alpha_2+1}$, *then:*

$$\lim_{N\to\infty}\Big[J_{z_k\bar{z}_l}\big(L, n\big) - J_{z_k\bar{z}_l}\big(M, N-n\big)\Big] = 1. \tag{A.2.31}$$

Appendix B

Correlation function asymptotics

Definition B.0.8. *The generalized limiting real Ginibre correlation kernel for the limiting mean density $\rho(z)$ is given by:*

$$\operatorname{Pfaff}\begin{bmatrix} K^{Generalized}(r,r') & K^{Generalized}(r,s') \\ K^{Generalized}(s,r') & K^{Generalized}(s,s') \end{bmatrix}. \tag{B.0.1}$$

1. *The limiting real/real kernel is given by:*

$$K^{Generalized}(r,r') = \pi\rho(u)\frac{\sqrt{\pi\rho(u)}}{\sqrt{2\pi}} \times \tag{B.0.2}$$
$$\begin{bmatrix} (r'-r)\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-r')^2} & \mathrm{e}^{-\sqrt{\pi\rho(u)}\frac{1}{2}(r-r')^2} \\ -\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-r')^2} & \sqrt{\frac{\pi}{2}}\operatorname{sgn}(r-r')\operatorname{erfc}\left(\sqrt{\pi\rho(u)}\frac{|r-r'|}{\sqrt{2}}\right) \end{bmatrix}.$$

2. *The limiting complex/complex kernel is given by:*

$$\begin{aligned} &K^{Generalized}(s,s') \\ =&\pi\rho(u)\frac{\sqrt{\pi\rho(u)}}{\sqrt{2\pi}}\Big(\operatorname{erfc}(\sqrt{2\pi\rho(u)}\operatorname{Im}(s))\operatorname{erfc}(\sqrt{2\pi\rho(u)}\operatorname{Im}(s'))\Big)^{\frac{1}{2}} \times \\ &\begin{bmatrix} (s'-s)\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(s-s')^2} & i(\bar{s}-s')\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(s-\bar{s}')^2} \\ i(s'-\bar{s})\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(\bar{s}-s')^2} & (\bar{s}-\bar{s}')\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(\bar{s}-\bar{s}')^2} \end{bmatrix}. \end{aligned} \tag{B.0.3}$$

3. *The limiting real/complex kernel is given by:*

$$K^{Generalized}(r,s) = \pi\rho(u)\frac{\sqrt{\pi\rho(u)}}{\sqrt{2\pi}}\big(\operatorname{erfc}(\sqrt{2\pi\rho(u)}\operatorname{Im}(s))\big)^{\frac{1}{2}} \times$$
$$\begin{bmatrix} (s-r)\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-s)^2} & i(\bar{s}-r)\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-\bar{s})^2} \\ -\,\mathrm{e}^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-s)^2} & -ie^{-\frac{1}{2}\sqrt{\pi\rho(u)}(r-\bar{s})^2} \end{bmatrix}. \tag{B.0.4}$$

Remark B.0.9. *Note that choosing* $\rho(u) = \frac{1}{\pi}\Theta(1-|u|)$ *gives limiting correlation kernel of the real Ginibre ensemble from [BS09], appendix A.*

Definition B.0.10. *The almost square limiting correlation kernel is given by:*

$$\operatorname{Pfaff}\begin{bmatrix} K_{origin}^{Generalized}(r,r') & K_{origin}^{Generalized}(r,s') \\ K_{origin}^{Generalized}(s,r') & K_{origin}^{Generalized}(s,s') \end{bmatrix}. \tag{B.0.5}$$

1. *The limiting real/real kernel is given by the* 2×2 *matrix:*

$$\begin{aligned} &K_{origin}^{Generalized}(r,r') \\ =&\frac{1}{\sqrt{2\pi}}\begin{bmatrix} (r'-r)\,\mathrm{e}^{-\frac{1}{2}(r-r')^2}\,\frac{\gamma(L,rr')}{\Gamma(L)} & \mathrm{e}^{-\frac{1}{2}(r-r')^2}\,\frac{\gamma(L,rr')}{\Gamma(L)} + t^{IndGin}(r,r') \\ -\,\mathrm{e}^{-\frac{1}{2}(r-r')^2}\,\frac{\gamma(L,rr')}{\Gamma(L)} - t^{IndGin}(r,r') & (*) \end{bmatrix} \end{aligned} \tag{B.0.6}$$

where

$$\begin{aligned} (*) =& -\frac{\gamma(L,r'^2)}{\Gamma(L)} + \mathrm{e}^{-\frac{1}{2}(r-r')^2}\,\frac{\gamma(L,rr')}{\Gamma(L)} \\ &+ \Big(\frac{r'^L\,\mathrm{e}^{\frac{1}{2}r'^2}}{\Gamma(L)} - 2^{\frac{L}{2}-1}\frac{\Gamma\big(\frac{L}{2},\frac{1}{2}r'^2\big)}{\Gamma\big(\frac{L}{2}\big)}\Big)\int_x^y \mathrm{e}^{\frac{1}{2}t}\,t^L dt. \end{aligned} \tag{B.0.7}$$

2. *The limiting complex/complex kernel is given by the* 2×2 *matrix:*

$$\begin{aligned} &K_{origin}^{Generalized}(s,s') = \frac{1}{\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\,\mathrm{Im}(s))\operatorname{erfc}(\sqrt{2}\,\mathrm{Im}(s'))}\times \\ &\begin{bmatrix} (s-s')\,\mathrm{e}^{-\frac{1}{2}(s-s')^2}\,\frac{\gamma(L,ss')}{\Gamma(L)} & i(\bar{s}-s')\,\mathrm{e}^{-\frac{1}{2}(s-\bar{s}')^2}\,\frac{\gamma(L,z\bar{z}')}{\Gamma(L)} \\ i(s'-\bar{s})\,\mathrm{e}^{-\frac{1}{2}(\bar{s}-s')^2}\,\frac{\gamma(L,\bar{s}s')}{\Gamma(L)} & (\bar{s}-\bar{s}')\,\mathrm{e}^{-\frac{1}{2}(\bar{s}-\bar{s}')^2}\,\frac{\gamma(L,\bar{s}\bar{s}')}{\Gamma(L)} \end{bmatrix}. \end{aligned} \tag{B.0.8}$$

3. *The limiting real/complex kernel is given by the* 2×2 *matrix:*

$$\begin{aligned} &K_{origin}^{Generalized}(r,s) = \frac{1}{\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\,\mathrm{Im}(z))}\times \\ &\begin{bmatrix} (s-r)\,\mathrm{e}^{-\frac{1}{2}(r-s)^2}\,\frac{\gamma(L,rs)}{\Gamma(L)} & i(\bar{s}-r)\,\mathrm{e}^{-\frac{1}{2}(r-\bar{s})^2}\,\frac{\gamma(L,r\bar{s})}{\Gamma(L)} \\ -\,\mathrm{e}^{-\frac{1}{2}(r-s)^2}\,\frac{\gamma(L,r\bar{s})}{\Gamma(L)} & -i\,\mathrm{e}^{-\frac{1}{2}(r-\bar{s})^2}\,\frac{\gamma(L,r\bar{s})}{\Gamma(L)} - it^{IndGin}(r,\bar{s}) \end{bmatrix}. \end{aligned} \tag{B.0.9}$$

B.1 Correlation function asymptotics for the real induced Ginibre ensemble

Theorem B.1.1 (Strong rectangularity in the bulk)**.** *Let* $u \in \mathbb{R}$ *such that* $\sqrt{\alpha} < |u| < \sqrt{\alpha+1}$ *and let* $r_1,\ldots,r_{K'} \in \mathbb{R}$ *as well as* $s_1,\ldots,s_{L'} \in \mathbb{C}_+\backslash\mathbb{R}$*. Furthermore set* $x_j = \sqrt{N}u + r_j$ *for* $j = 1,\ldots,K'$*,* $z_m = \sqrt{N}u + s_m$ *for* $m = 1,\ldots,L'$ *and*

$L = N\alpha$, then:

$$\begin{aligned}
&\lim_{N\to\infty} R^{IndGin}_{K',L'}(x_1,\dots,x_{K'},z_1,\dots,z_{L'})\\
&= \mathrm{Pfaff}\begin{bmatrix} K^{Generalized}(r_t,r_{t'}) & K^{Generalized}(r_t,s_{v'}) \\ K^{Generalized}(s_v,r_{t'}) & K^{Generalized}(s_v,s_{v'}) \end{bmatrix},
\end{aligned} \tag{B.1.1}$$

where $t,t' = 1,\dots,K'$ and $v,v' = 1,\dots,L'$ and

$$\rho(u) = \frac{1}{\pi}\big[\Theta\big(|u|-\sqrt{\alpha}\,\big) - \Theta\big(|u|-\sqrt{\alpha+1}\,\big)\big]. \tag{B.1.2}$$

Proof. The complex-complex correlation kernel is given by:

$$\begin{aligned}
DS^{\mathrm{IndGin}}_N(z_j,z_{j'}) =&(z_{j'}-z_j)s_N(z_j,z_{j'})\\
S^{\mathrm{IndGin}}_N(z_j,z_{j'}) =&i(\bar z_{j'}-z_j)s_N(z_j,\bar z_{j'})\\
IS^{\mathrm{IndGin}}_N(\bar z_j,\bar z_{j'}) =&(\bar z_j-\bar z_{j'})s_N(\bar z_j,\bar z_{j'}),
\end{aligned}$$

where

$$\begin{aligned}
s^{\mathrm{IndGin}}_N(z_j,z_{j'}) =&\frac{1}{\sqrt{2\pi}}\Big(\operatorname{erfc}\big(\sqrt{2}\,\mathrm{Im}(z_j)\big)\operatorname{erfc}\big(\sqrt{2}\,\mathrm{Im}(z_{j'})\big)\Big)^{-\frac{1}{2}}\times\\
&\mathrm{e}^{-\frac14 z_j^2-\frac14\bar z_j^2-\frac14 z_{j'}^2-\frac14\bar z_{j'}^2+z_jz_{j'}}\left[\frac{\gamma(L,z_jz_{j'})}{\Gamma(L)}-\frac{\gamma(L+N,z_jz_{j'})}{\Gamma(L+N)}\right].
\end{aligned} \tag{B.1.3}$$

Theorem A.1.3 then gives:

$$\left[\frac{\gamma(L,z_jz_{j'})}{\Gamma(L)}-\frac{\gamma(L+N,z_jz_{j'})}{\Gamma(L+N)}\right]\sim 1. \tag{B.1.4}$$

Moreover:

$$\mathrm{e}^{-\frac14 z_j^2-\frac14\bar z_j^2-\frac14 z_{j'}^2-\frac14\bar z_{j'}^2+z_jz_{j'}}\sim \mathrm{e}^{-\frac12(s_j-s_{j'})^2}. \tag{B.1.5}$$

All in all:

$$s^{\mathrm{IndGin}}_N(z_j,z_{j'}) \sim\Big(\operatorname{erfc}\big(\sqrt{2}\,\mathrm{Im}(s_j)\big)\operatorname{erfc}\big(\sqrt{2}\,\mathrm{Im}(s_{j'})\big)\Big)^{-\frac{1}{2}}\mathrm{e}^{-\frac12(s_j-s_{j'})^2}. \tag{B.1.6}$$

and the complex kernel entry expression can be easily deduced.

Next we move to the derivation of the real/complex kernel. We need:

Lemma B.1.2. *Let $u \in \mathbb{R}$ with $\sqrt{\alpha} < u < \sqrt{\alpha+1}$ and $s \in \mathbb{C}$, $r \in \mathbb{C}$. Then*

$$\lim_{N\to\infty} r_N^{IndGin}(\sqrt{N}u + r, \sqrt{N}u + s) = 0 \tag{B.1.7}$$

Proof.

$$r_N^{\mathrm{IndGin}}(\sqrt{N}u + r, \sqrt{N}u + s) = \frac{1}{\sqrt{2\pi}} \operatorname{sgn}(\sqrt{N}u + r) 2^{\frac{N}{2}(\alpha+1)-\frac{3}{2}} (\sqrt{N}u + s)^{N(\alpha+1)-1}$$
$$\times \left(\operatorname{erfc}\left(\sqrt{2}\operatorname{Im}(s)\right)\right)^{\frac{1}{2}} e^{-\frac{1}{4}(\sqrt{N}u+s)^2 - \frac{1}{4}(\sqrt{N}u+\bar{s})^2} \frac{\gamma\left(\frac{N}{2}(\alpha+1) - \frac{1}{2}, \frac{1}{2}(\sqrt{N}u + r^2)\right)}{\Gamma(N(\alpha+1)-1)}$$

We can apply the duplication formula for the gamma function:

$$\Gamma(2z) = \Gamma(z)\Gamma(z+\frac{1}{2})2^{2z-1}\frac{1}{\sqrt{\pi}}, \tag{B.1.8}$$

and obtain:

$$\begin{aligned} &r_N^{\mathrm{IndGin}}(\sqrt{N}u + r, \sqrt{N}u + s) \qquad\qquad (\text{B.1.9}) \\ &= \operatorname{sgn}(\sqrt{N}u + r)\left(\operatorname{erfc}\left(\sqrt{2}\operatorname{Im}(s)\right)\right)^{\frac{1}{2}}\left(1 + \frac{r'}{\sqrt{N}u}\right)^{N(\alpha+1)-1} \times \\ &\left(\frac{N}{2}u^2\right)^{\frac{N}{2}(\alpha+1)-\frac{1}{2}} e^{-\frac{1}{2}Nu^2 - \sqrt{N}u\operatorname{Re}(s) - \operatorname{Re}(s)^2 + \operatorname{Im}(s)^2} \frac{\gamma\left(\frac{N}{2}(\alpha+1) - \frac{1}{2}, \frac{1}{2}(\sqrt{N}u + r^2)\right)}{\Gamma(\frac{N}{2}(\alpha+1))\Gamma(\frac{N}{2}(\alpha+1) - \frac{1}{2})}. \end{aligned}$$

Furthermore the use of the Stirling formula, as well as:

$$\left(1 + \frac{s}{\sqrt{N}u}\right)^{N(\alpha+1)-1} \sim e^{\sqrt{N}(\alpha+1)\frac{s}{u} - \frac{s^2}{2u^2}(\alpha+1)^2}. \tag{B.1.10}$$

leads to:

$$\begin{aligned} &r_N^{\mathrm{IndGin}}(\sqrt{N}u + r, \sqrt{N}u + s) \qquad\qquad (\text{B.1.11}) \\ &\sim \frac{1}{\sqrt{\pi}} \operatorname{sgn}(u)\left(\operatorname{erfc}\left(\sqrt{2}\operatorname{Im}(s)\right)\right)^{\frac{1}{2}} \left(\frac{u^2}{\alpha+1}\right)^{\frac{N}{2}(\alpha+1)} \mathrm{e}^{\sqrt{N}(\alpha+1)\frac{s}{u} - \frac{s^2}{2u^2}(\alpha+1)^2 + \frac{N}{2}(\alpha+1)} \times \\ &\mathrm{e}^{-\frac{1}{2}Nu^2 - \sqrt{N}u\operatorname{Re}(s) - \operatorname{Re}(s)^2 + \operatorname{Im}(s)^2} \frac{\gamma\left(\frac{N}{2}(\alpha+1) - \frac{1}{2}, \frac{1}{2}(\sqrt{N}u + r^2)\right)}{\Gamma(\frac{N}{2}(\alpha+1-\frac{1}{2})}. \end{aligned}$$

It can easily be shown that:

$$e^{\sqrt{N}(\alpha+1)\frac{s}{u} - \frac{s^2}{2u^2}(\alpha+1)^2 + \frac{N}{2}(\alpha+1) - \frac{1}{2}Nu^2 - \sqrt{N}u\operatorname{Re}(s)} \sim 1. \tag{B.1.12}$$

and thus using A.1.1 proves our result. □

Additionally we need

Lemma B.1.3. *Let $u \in \mathbb{R}$ with $\sqrt{\alpha} < u < \sqrt{\alpha+1}$ and $s \in \mathbb{C}$, $r \in \mathbb{C}$. Then:*

$$\lim_{N\to\infty} t^{IndGin}(\sqrt{N}u + r, \sqrt{N}u + s) = 0. \tag{B.1.13}$$

Proof. Noting that:

$$\begin{aligned} &t^{\mathrm{IndGin}}(\sqrt{N}u + r, \sqrt{N}u + s) \\ =&\frac{1}{\sqrt{2\pi}} \frac{2^{\frac{N\alpha}{2}-1}}{\Gamma(N\alpha)} e^{-\frac{1}{2}(\sqrt{N}u+r')^2} (\sqrt{N}u + r')^{N\alpha} \Gamma(\frac{N}{2}\alpha, \frac{1}{2}(\sqrt{N}u + r)^2), \end{aligned} \tag{B.1.14}$$

we proceed exactly as in B.1.2. ☐

Combining the asymptotics behavior of s_N^{IndGin} with B.1.2 and B.1.3, then gives the scaling limit of real/complex kernel. The limiting behavior of the real/real correlation kernel entries S_N^{IndGin} and DS_N^{IndGin} follow as well. It remains to determine the scaling limit for the entry IS_N^{IndGin}. Employing the saddle point method on each of the eight integrals defining IS_N^{IndGin} yields the desired result. ☐

Theorem B.1.4 (Strong Rectangularity in the complex bulk)**.** *Let u be a complex number such that $\sqrt{\alpha} < |u| < \sqrt{\alpha+1}$ and let $s_1, \ldots, s_{L'} \in \mathbb{C}$. Furthermore set $z_m = \sqrt{N}u + s_m$ for $j = 1, \ldots, L'$ and $L = N\alpha$, then:*

$$\lim_{N\to\infty} R^{IndGin}_{0,L'}(-, z_1, \ldots, z_{L'}) = \det\left(\frac{1}{\pi}\, \mathrm{e}^{-\frac{|s_j|}{2} - \frac{|s_{j'}|}{2} + s_j \bar{s}_{j'}}\right)_{j,j'=1}^{L'}. \tag{B.1.15}$$

Proof. The complex-complex correlations are given by:

$$R^{\mathrm{IndGin}}_{0,L'}(-, z_1, \ldots, z_{L'}) = \mathrm{Pfaff}\begin{bmatrix} DS_N(z_j, z_{j'}) & S_N(z_j, z_{j'}) \\ -S_N(z_j, z_{j'}) & IS_N(z_j, z_{j'}) \end{bmatrix} \tag{B.1.16}$$

where

$$\begin{aligned} DS_N^{\mathrm{IndGin}}(z_j, z_{j'}) =&(z_{j'} - z_j) s_N(z_j, z_{j'}) \\ S_N^{\mathrm{IndGin}}(z_j, z_{j'}) =&i(\bar{z}_{j'} - z_j) s_N(z_j, \bar{z}_{j'}) \\ IS_N^{\mathrm{IndGin}}(\bar{z}_j, \bar{z}_{j'}) =&(\bar{z}_j - \bar{z}_{j'}) s_N(\bar{z}_j, \bar{z}_{j'}) \end{aligned}$$

and

$$\begin{aligned} s_N^{\mathrm{IndGin}}(z_j, z_{j'}) =&\frac{1}{\sqrt{2\pi}} \Big(\mathrm{erfc}\left(\sqrt{2}\,\mathrm{Im}(z_j)\right) \mathrm{erfc}\left(\sqrt{2}\,\mathrm{Im}(z_{j'})\right)\Big)^{-\frac{1}{2}} \times \\ &\mathrm{e}^{-\frac{1}{4}z_j^2 - \frac{1}{4}\bar{z}_j^2 - \frac{1}{4}z_{j'}^2 - \frac{1}{4}\bar{z}_{j'}^2 + z_j z_{j'}} \left[\frac{\gamma(L, z_j z_{j'})}{\Gamma(L)} - \frac{\gamma(L+N, z_j z_{j'})}{\Gamma(L+N)}\right] \end{aligned} \tag{B.1.17}$$

s_N^{IndGin} is already in a convenient form for the asymptotic analysis and we can immediately apply theorem A.1.3, which gives:

$$\left[\frac{\gamma(L, z_j z_{j'})}{\Gamma(L)} - \frac{\gamma(L+N, z_j z_{j'})}{\Gamma(L+N)}\right] \sim 1. \tag{B.1.18}$$

Furthermore from [AS72], page 298, equation (7.1.23):

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2}, z^2) \sim \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}|z|}. \tag{B.1.19}$$

and thus:

$$\begin{aligned}\left(\operatorname{erfc}\left(\sqrt{2}\operatorname{Im}(z_j)\right)\operatorname{erfc}\left(\sqrt{2}\operatorname{Im}(z_{j'})\right)\right)^{-\frac{1}{2}} &\sim \frac{\mathrm{e}^{-2\operatorname{Im}(z_j)^2-2\operatorname{Im}(z_{j'})^2}}{\sqrt{\pi|z_{j'}z_j|}} \\ \sim &\frac{\mathrm{e}^{-4N\operatorname{Im}(u)^2-4\sqrt{N}\operatorname{Im}(u)^2(\operatorname{Im}(s_j)+\operatorname{Im}(s_{j'}))-\operatorname{Im}(s_j)^2-\operatorname{Im}(s_{j'})^2}}{\sqrt{2N\pi}\operatorname{Im}(u)}.\end{aligned} \tag{B.1.20}$$

In addition:

$$\mathrm{e}^{-\frac{1}{4}z_j^2-\frac{1}{4}\bar{z}_j^2-\frac{1}{4}z_{j'}^2-\frac{1}{4}\bar{z}_{j'}^2+z_jz_{j'}} \sim \mathrm{e}^{-\frac{1}{2}(s_j-s_{j'})^2} \tag{B.1.21}$$

All in all:

$$\begin{aligned}DS_N^{\mathrm{IndGin}}(z_j, z_{j'}) \sim &\frac{1}{\sqrt{2\pi}}(s_j - s_{j'})\,\mathrm{e}^{-\frac{1}{2}(s_j-s_{j'})^2-\operatorname{Im}(s_j)^2-\operatorname{Im}(s_{j'})^2}\,\frac{1}{\sqrt{2N\pi}\operatorname{Im}(u)} \times \\ &\mathrm{e}^{-4N\operatorname{Im}(u)^2-4\sqrt{N}\operatorname{Im}(u)^2(\operatorname{Im}(s_j)+\operatorname{Im}(s_{j'}))}.\end{aligned} \tag{B.1.22}$$

Using:

$$\mathrm{e}^{-4N\operatorname{Im}(u)^2-4\sqrt{N}\operatorname{Im}(u)^2(\operatorname{Im}(s_j)+\operatorname{Im}(s_{j'}))} \sim 1. \tag{B.1.23}$$

It follows that:

$$\lim_{N\to\infty} DS_N^{\mathrm{IndGin}}(z_j, z_{j'}) = 0, \tag{B.1.24}$$

which implies:

$$\lim_{N\to\infty} IS_N^{\mathrm{IndGin}}(z_j, z_{j'}) = 0 \tag{B.1.25}$$

due to the relation $DS_N^{\mathrm{IndGin}}(z_j, z_{j'}) = -IS_N^{\mathrm{IndGin}}(z_j, z_{j'})$. Furthermore:

$$\begin{aligned}S_N^{\mathrm{IndGin}}(z_j, z_{j'}) \sim &\frac{i}{\sqrt{2\pi}}\frac{(-2i\sqrt{N}\operatorname{Im}(u)\bar{s}_{j'} - s_j)}{\sqrt{2N\pi}\operatorname{Im}(u)}\,\mathrm{e}^{-\frac{1}{2}(s_j-s_{j'})^2-\operatorname{Im}(s_j)^2-\operatorname{Im}(s_{j'})^2} \\ \sim &\frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|s_j|^2-\frac{1}{2}|s_{j'}|^2+s_j\bar{s}_{j'}}.\end{aligned} \tag{B.1.26}$$

As a consequence:

$$\begin{aligned}&\lim_{N\to\infty} R_{0,L'}^{\mathrm{IndGin}}(-,z_1,\ldots,z_{L'})\\ &=\mathrm{Pfaff}\left[\begin{array}{cc}0 & \frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|s_j|^2-\frac{1}{2}|s_{j'}|^2+s_j\bar{s}_{j'}}\\ -\frac{1}{\pi}\,\mathrm{e}^{-\frac{1}{2}|s_j|^2-\frac{1}{2}|s_{j'}|^2+s_j\bar{s}_{j'}} & 0\end{array}\right]_{j,j'=1}^{L'}.\end{aligned} \tag{B.1.27}$$

$\square$

Theorem B.1.5 (Strong rectangularity at the edges)**.** *Let $u=\pm1$, $r_1,\ldots,r_{K'}\in\mathbb{R}$ as well as $s_1,\ldots,s_{L'}\in\mathbb{C}_+$. Setting $x_j^{out}=\sqrt{N(\alpha+1)}u+r_j$ for $j=1,\ldots,K'$ and $z_m^{out}=\sqrt{N(\alpha+1)}u+s_m$ for $m=1,\ldots,L'$ leads to Ginibre behavior for the limiting correlation functions at the outer edge of the eigenvalue distribution as described in [BS09]. In addition at the inner circular edge $x_j^{in}=\sqrt{N\alpha}u-r_j$ for $t=1,\ldots,K'$ and $z_m^{in}=\sqrt{N\alpha}u-s_m$ for $m=1,\ldots,L'$ the Ginibre limiting correlation functions can again be recovered. More precisely*

$$\begin{aligned}&\lim_{N\to\infty} R_{K',L'}^{IndGin}(x_1^{out},\ldots,x_{K'}^{out},z_1^{out},\ldots,z_{L'}^{out})=\lim_{N\to\infty} R_{K',L'}^{IndGin}(x_1^{in},\ldots,x_{K'}^{in},z_1^{in},\ldots,z_{L'}^{in})\\ &=\mathrm{Pfaff}\left[\begin{array}{cc}K_{edge}^{IndGin}(r_j,r_{j'}) & K_{edge}^{IndGin}(r_j,s_{m'})\\ K_{edge}^{IndGin}(s_m,r_{j'}) & K_{edge}^{IndGin}(s_m,s_{m'})\end{array}\right]\end{aligned} \tag{B.1.28}$$

where $j,j'=1,\ldots,K'$ and $m,m'=1,\ldots,L'$.

1. *The entries of the limiting real/real kernel are given by:*

$$\begin{aligned}S_{edge}^{IndGin}(r,r') &= \frac{1}{2\sqrt{2\pi}}e^{-\frac{1}{2}(r-r')^2}\,\mathrm{erfc}\left(u\frac{r+r'}{\sqrt{2}}\right)+\frac{1}{4\sqrt{\pi}}\,\mathrm{e}^{-r^2}\,\mathrm{erfc}(ur')\\ DS_{edge}^{IndGin}(r,r') &= \frac{1}{2\sqrt{2\pi}}(r'-r)e^{-\frac{1}{2}(r-r')^2}\,\mathrm{erfc}\left(u\frac{r+r'}{\sqrt{2}}\right)\\ IS_{edge}^{IndGin}(r,r') &= \frac{1}{2}\,\mathrm{sgn}(r-r')\,\mathrm{erfc}\left(\frac{|r-r'|}{\sqrt{2}}\right)\end{aligned}$$

2. *The entries of the limiting complex/complex kernel are given by:*

$$
\begin{aligned}
S^{IndGin}_{edge}(s,s') &= \frac{i}{2\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s'))}\times \\
&\quad (\bar{s}'-s)e^{-\frac{1}{2}(s-\bar{s}')^2}\operatorname{erfc}\left(u\frac{s+\bar{s}'}{\sqrt{2}}\right) \\
DS^{IndGin}_{edge}(s,s') &= \frac{1}{2\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s'))}\times \\
&\quad (s'-s)e^{-\frac{1}{2}(s-s')^2}\operatorname{erfc}\left(u\frac{s+s'}{\sqrt{2}}\right) \\
IS^{IndGin}_{edge}(s,s') &= \frac{1}{\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s'))}\times \\
&\quad (\bar{s}-\bar{s}')e^{-\frac{1}{2}(\bar{s}-\bar{s}')^2}\operatorname{erfc}\left(u\frac{\bar{s}+\bar{s}'}{\sqrt{2}}\right)
\end{aligned}
$$

3. *The entries of the limiting real/complex kernel are given :*

$$
\begin{aligned}
S^{IndGin}_{edge}(r,s) &= \frac{i}{2\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))}(\bar{s}-r)e^{-\frac{1}{2}(r-\bar{s})^2}\operatorname{erfc}\left(u\frac{r+\bar{s}}{\sqrt{2}}\right) \\
S^{IndGin}_{edge}(s,r) &= \frac{1}{2\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))}(\bar{s}-r)e^{-\frac{1}{2}(r-s)^2}\operatorname{erfc}\left(u\frac{r+s}{\sqrt{2}}\right) \\
&\quad + \frac{1}{4\sqrt{\pi}}e^{-s^2}\operatorname{erfc}(ur) \\
DS^{IndGin}_{edge}(s,r) &= \frac{1}{2\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))}(s-r)e^{-\frac{1}{2}(s-r)^2}\operatorname{erfc}\left(u\frac{s+r}{\sqrt{2}}\right) \\
IS^{IndGin}_{edge}(s,s') &= \frac{-i}{\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))}e^{-\frac{1}{2}(r-\bar{s})^2}\operatorname{erfc}\left(u\frac{r+\bar{s}}{\sqrt{2}}\right) \\
&\quad - \frac{i}{4\sqrt{\pi}}e^{-\bar{s}^2}\operatorname{erfc}(ur)
\end{aligned}
$$

Proof. We start with the complex/complex kernel entries and note first that:

$$
\begin{aligned}
&s^{\mathrm{IndGin}}_N(\sqrt{N(\alpha+1)}u+s,\sqrt{N(\alpha+1)}u+s') \\
=&\frac{1}{\sqrt{2\pi}}\sqrt{\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s))\operatorname{erfc}(\sqrt{2}\operatorname{Im}(s'))}\,\mathrm{e}^{-\frac{1}{2}(s-s')^2}\left[\frac{\gamma(L,zz')}{\Gamma(L)}-\frac{\gamma(L+N,zz')}{\Gamma(L+N)}\right]
\end{aligned}
$$

Applying A.1.4 gives the scaling limit at the edge for all the complex/complex entries. In the following we need

Lemma B.1.6. *Let $u=\pm 1$ as well as $s\in\mathbb{C}^+$, $r\in\mathbb{R}$. In addition set $z^{out}=\sqrt{N(\alpha+1)}u+s$, $x^{out}=\sqrt{N(\alpha+1)}u+r$ as well as $z^{in}=\sqrt{N\alpha}u+s$, $x^{in}=$*

$\sqrt{N\alpha}u + r$. *Then*

$$\lim_{N\to\infty} r_N^{IndGin}(x^{out}, z^{out}) = \lim_{N\to\infty} t^{IndGin}(x^{in}, z^{in}) = \frac{1}{4\sqrt{\pi}} e^{-s^2} \operatorname{erfc}(ur) \tag{B.1.29}$$

$$\lim_{N\to\infty} r_N^{IndGin}(x^{in}, z^{in}) = \lim_{N\to\infty} t^{IndGin}(x^{out}, z^{out}) = 0 \tag{B.1.30}$$

Proof. We will just prove the relations for r_N^{IndGin} as the relations for t^{IndGin} follow analogously. From the proof of theorem **??**

$$r_N^{\text{IndGin}}(x^{\text{out}}, z^{\text{out}}) \sim \frac{1}{2\sqrt{\pi}} e^{-s^2} \frac{\Gamma\Big(\frac{(N(\alpha+1)-1}{2}, N(\alpha+1) - \sqrt{N(\alpha+1)}(s+r) + sr\Big)}{\Gamma\big(\frac{N(\alpha+1)-1}{2}\big)}$$

$$r_N^{\text{IndGin}}(x^{\text{in}}, z^{\text{in}}) \sim \frac{1}{2\sqrt{\pi}} e^{-s^2} \frac{\Gamma\Big(\frac{(N(\alpha+1)-1}{2}, N\alpha - \sqrt{N\alpha}(s+r) + sr\Big)}{\Gamma\big(\frac{N(\alpha+1)-1}{2}\big)}$$

Applying A.1.2 then gives the desired result. □

The scaling limits now follow using the definition of the kernel entries. □

Theorem B.1.7 (The limiting correlation functions at the complex edges)**.** *Let $u, s_1, \ldots, s_m$ be complex numbers with $|u| = 1$, setting $z_k = \sqrt{N(\alpha+1)}u + s_k$ for $k = 1, \ldots, m$ leads to the limiting correlation functions at the outer edge $r_{out} = \sqrt{L+N}$:*

$$\lim_{N\to\infty} R_{0,m}^{IndGin}(-, z_1, \ldots, z_m) = \det\left[\frac{1}{\pi} \mathrm{e}^{-\frac{1}{2}|s_j|^2 - \frac{1}{2}|s_{j'}|^2 + s_j \bar{s}_{j'}} \operatorname{erfc}\left(\frac{s_j \bar{u} + \bar{s}_{j'} u}{\sqrt{2}}\right)\right]_{j,j'=1}^{n}.$$

The same limiting expression is found around the inner edge $r_{in} = \sqrt{L}$ of the eigenvalue density by setting $z_k = \sqrt{N\alpha}u - s_k$ for $k = 1, \ldots, m$.

Proof. Combine the first part of the proof of B.1.4 with theorem A.1.4. □

B.2 Correlation function asymptotics for the real induced spherical ensemble

Theorem B.2.1 (Strong rectangularity and strong spherical component)**.** *Let A be a matrix pertaining to the real induced spherical ensemble, in the regime of strong rectangularity and strong spherical component: $L = N\alpha$ and $n - N = N\beta$. Set $\frac{\alpha}{\beta+1} := \mu_1$ and $\frac{\alpha+1}{\beta} =: \mu_2$. In addition let $u \in \mathbb{R}$ such that $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$ and let $r_1, \ldots, r_j \in \mathbb{R}$ as well as $s_1, \ldots, s_m \in \mathbb{C}_+ \backslash \mathbb{R}$. Furthermore set $x_j =$*

$u + \frac{r_j}{\sqrt{n+L}}$ *for* $j = 1, \ldots, K'$, $z_m = u + \frac{s_m}{\sqrt{n+L}}$ *for* $m = 1, \ldots, L'$, *then:*

$$\lim_{N\to\infty} \frac{1}{(n+L)^N} R^{IndSpherical}_{K',L'}(x_1, \ldots, x_{K'}, z_1, \ldots, z_{L'})$$
$$= \text{Pfaff} \begin{bmatrix} K^{Generalized}(r_j, r_{j'}) & K^{Generalized}(r_j, s_{m'}) \\ K^{Generalized}(s_m, r_{j'}) & K^{Generalized}(s_m, s_{m'}) \end{bmatrix}, \tag{B.2.1}$$

with $j, j' = 1, \ldots, K'$, $m, m' = 1, \ldots, L'$ *and:*

$$\rho(u) := \frac{1}{\pi} \frac{1}{(1+u^2)^2} \big[\Theta(|u| - \sqrt{\mu_1}) - \Theta(|u| - \sqrt{\mu_2})\big]. \tag{B.2.2}$$

Proof. The real correlation kernels are completely defined through the functions s_N, r_N, t from (4.2.105)–(4.2.109). Thus starting point of the asymptotic analysis is the integral representation:

$$s_N^{\text{IndSpherical}}(z_j, z_{j'})$$
$$= \frac{2(n+L)(n+L-1)}{\pi} \Big(\int_{\frac{2|\operatorname{Im}(z_j)|}{|1+z_j^2|}}^{\infty} (1+t^2)^{\frac{n+L+1}{2}} dt \int_{\frac{2|\operatorname{Im}(z_{j'})|}{|1+z_{j'}^2|}}^{\infty} (1+t^2)^{\frac{n+L+1}{2}} dt \Big)^{\frac{1}{2}} \times$$
$$\frac{(1+z_j z_{j'})^{n+L-2}}{\big(|1+z_j^2||1+z_{j'}^2|\big)^{\frac{n+L+1}{2}}} \Big[I_{\frac{z_j z_{j'}}{1+z_j z_{j'}}}(L, n-1) - I_{\frac{z_j z_{j'}}{1+z_j z_{j'}}}(M-1, n-N) \Big]. \tag{B.2.3}$$

From appendix A we can apply A.2.6, which gives:

$$I_{\frac{z_j z_{j'}}{1+z_j z_{j'}}}(L, n-1) - I_{\frac{z_j z_{j'}}{1+z_j z_{j'}}}(M-1, n-N) \sim 1. \tag{B.2.4}$$

Furthermore as $\frac{2|\operatorname{Im}(z_j)|}{|1+z_j^2|} \sim \frac{2|\operatorname{Im}(s_j)|}{\sqrt{n+L}(1+u^2)}$:

$$\int_{\frac{2|\operatorname{Im}(z_j)|}{|1+z_j^2|}}^{\infty} (1+t^2)^{\frac{n+L+1}{2}} dt$$
$$\sim \frac{1}{\sqrt{n+L}} \int_{\frac{2|\operatorname{Im}(s_j)|}{1+u^2|}}^{\infty} e^{-\frac{1}{2}t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{2(n+L)}} \operatorname{erfc}\Big(\sqrt{2}\frac{|\operatorname{Im}(s_j)|}{1+u^2}\Big). \tag{B.2.5}$$

In addition note:

$$|1+z_j^2|^{-\frac{n+L+1}{2}} \sim (1+u^2)^{-\frac{n+L+1}{2}} e^{-\frac{1}{2}\frac{s_j^2}{1+u^2}} \tag{B.2.6}$$
$$(1+z_j z_{j'})^{n+L-2} \sim (1+u^2)^{n+L-2} e^{-\frac{1}{2}\frac{s_j s_{j'}}{1+u^2}}. \tag{B.2.7}$$

All in all we derived the limiting complex-complex correlation kernel. Furthermore note that:

$$\frac{1}{B\big(\frac{n-N+1}{2},\frac{1}{2}\big)} \sim \sqrt{\frac{M}{2\pi}}(1+\mu_1)^{\frac{n+L+1}{2}}\mu_2^{-\frac{M-1}{2}} \tag{B.2.8}$$

$$z_j^M \sim u^{M-1}\,\mathrm{e}^{-\frac{1}{2}\frac{s_j^2}{1+u^2}}\,. \tag{B.2.9}$$

In addition A.2.1 gives:

$$I_{\frac{r_j^2}{1+r_j^2}}\Big(\frac{n+1}{2},\frac{n-N+2}{2}\Big) \sim \Theta(|u|-\sqrt{\mu_2}), \tag{B.2.10}$$

which in turn shows:

$$\lim_{N\to\infty}\frac{1}{n+L}r_N(r_j,z_j)=0. \tag{B.2.11}$$

Analogously we can show, that:

$$\lim_{N\to\infty}\frac{1}{n+L}t(r_j,z_j)=0. \tag{B.2.12}$$

The final missing limiting expression for kernel entry $IS_N^{\mathrm{IndSpherical}}(r_j,r_{j'})$ can be obtained by a saddle-point analysis on all eight integrals defining $IS_N^{\mathrm{IndSpherical}}$. $\square$

Theorem B.2.2 (Strong rectangularity and weak spherical component)**.** *Let A be a matrix pertaining to the real induced spherical ensemble, in the regime of strong rectangularity and weak spherical component: $L=N\alpha$ and $n-N=O(1)$. Furthermore set $\frac{L}{n-1}\sim\alpha:=\mu_1$. In addition let $u\in\mathbb{R}$ such that $\sqrt{\mu_1}<|u|$ and let $r_1,\dots,r_j\in\mathbb{R}$ as well as $s_1,\dots,s_m\in\mathbb{C}_+\backslash\mathbb{R}$. Furthermore set $x_j=u+\frac{r_j}{\sqrt{n+L}}$ for $j=1,\dots,K'$, $z_m=u+\frac{s_m}{\sqrt{n+L}}$ for $m=1,\dots,L'$, then:*

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{(n+L)^N}\,R_{K',L'}^{\mathit{IndSpherical}}(x_1,\dots,x_{K'},z_1,\dots,z_{L'})\\ &=\mathrm{Pfaff}\begin{bmatrix}K^{\mathit{Generalized}}(r_j,r_{j'}) & K^{\mathit{Generalized}}(r_j,s_{m'})\\ K^{\mathit{Generalized}}(s_m,r_{j'}) & K^{\mathit{Generalized}}(s_m,s_{m'})\end{bmatrix},\end{aligned} \tag{B.2.13}$$

with $j,j'=1,\dots,K'$, $m,m'=1,\dots,L'$ and:

$$\rho(u):=\frac{1}{\pi}\frac{1}{(1+u^2)^2}\Theta(|u|-\sqrt{\mu_1}). \tag{B.2.14}$$

Theorem B.2.3 (Almost square and strong spherical component)**.** *Let A be a matrix pertaining to the real induced spherical ensemble, in the regime of almost square matrices with strong spherical component: $L=O(1)$ and $n-N=N\beta$. Furthermore set $\frac{N+L-1}{N\beta}\sim\frac{1}{\beta}:=\mu_2$. In addition let $u\in\mathbb{R}$ such that $0<|u|<\sqrt{\mu_2}$*

and let $r_1,\ldots,r_j \in \mathbb{R}$ as well as $s_1,\ldots,s_m \in \mathbb{C}_+\backslash\mathbb{R}$. Furthermore set $x_j = u+\frac{r_j}{\sqrt{n+L}}$ for $j=1,\ldots,K'$, $z_m = u+\frac{s_m}{\sqrt{n+L}}$ for $m=1,\ldots,L'$, then in the bulk:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{(n+L)^N}\;R^{\mathit{IndSpherical}}_{K',L'}(x_1,\ldots,x_{K'},z_1,\ldots,z_{L'})\\ =&\operatorname{Pfaff}\begin{bmatrix} K^{\mathit{Generalized}}(r_j,r_{j'}) & K^{\mathit{Generalized}}(r_j,s_{m'})\\ K^{\mathit{Generalized}}(s_m,r_{j'}) & K^{\mathit{Generalized}}(s_m,s_{m'})\end{bmatrix},\end{aligned} \tag{B.2.15}$$

with $j,j'=1,\ldots,K'$, $m,m'=1,\ldots,L'$ and:

$$\rho(u) := \frac{1}{\pi}\frac{1}{(1+u^2)^2}\Theta(\sqrt{\mu_2}-|u|). \tag{B.2.16}$$

At the origin $u=0$:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{(n+L)^N}\;R^{\mathit{IndSpherical}}_{K',L'}(x_1,\ldots,x_{K'},z_1,\ldots,z_{L'})\\ =&\operatorname{Pfaff}\begin{bmatrix} K^{\mathit{Generalized}}_{\mathit{origin}}(r_j,r_{j'}) & K^{\mathit{Generalized}}_{\mathit{origin}}(r_j,s_{m'})\\ K^{\mathit{Generalized}}_{\mathit{origin}}(s_m,r_{j'}) & K^{\mathit{Generalized}}_{\mathit{origin}}(s_m,s_{m'})\end{bmatrix}.\end{aligned} \tag{B.2.17}$$

Proof. From the complex equivalent we know:

$$J_{\frac{s_js_{j'}}{n+L}}(L,n-1) - J_{\frac{s_js_{j'}}{n+L}}(M-1,n-N) \sim \frac{\gamma(s_js_{j'},L)}{\Gamma(L)}. \tag{B.2.18}$$

In addition note:

$$\frac{\left(1+\frac{s_js_{j'}}{n+L}\right)^{n+L-2}}{\left(|1+\frac{s_j^2}{n+L}||1+\frac{s_{j'}^2}{n+L}|\right)^{\frac{n+L+1}{2}}} \sim \mathrm{e}^{-\frac{1}{2}(s_j-s_{j'})^2}\,. \tag{B.2.19}$$

and from previous proof:

$$\int_{\frac{2|\operatorname{Im}(z_j)|}{|1+z_j^2|}}^{\infty}(1+t^2)^{\frac{n+L+1}{2}}dt \sim \frac{1}{2}\sqrt{\frac{\pi}{2(n+L)}}\operatorname{erfc}\left(\sqrt{2}\frac{|\operatorname{Im}(s_j)|}{1+u^2}\right). \tag{B.2.20}$$

Equally:

$$\lim_{N\to\infty}\frac{1}{n+L}\;r_N^{\mathrm{IndSpherical}}(r_j,z_j) = 0. \tag{B.2.21}$$

It remains to analyze:

$$t^{\mathrm{IndSpherical}}\left(\frac{r_j}{\sqrt{n+L}},\frac{s_j}{\sqrt{n+L}}\right) = \frac{1}{B\left(\frac{n}{2},\frac{L}{2}\right)}\frac{\left(\frac{s_j}{\sqrt{n+L}}\right)^L}{|1+\frac{s_j^2}{n+L}|^{\frac{n+L+1}{2}}}I_{\left(1+\frac{s_j^2}{n+L}\right)^{-1}}\left(\frac{n-1}{2},\frac{L}{2}\right).$$

Note that:

$$I_{\left(1+\frac{s_j^2}{n+L}\right)^{-1}}\Big(\frac{n-1}{2},\frac{L}{2}\Big)=1-J_{\frac{s_j^2}{n+L}}\Big(\frac{L}{2},\frac{n-1}{2}\Big)=\frac{1}{B\big(\frac{n-1}{2},\frac{L}{2}\big)}\int_{\frac{s_j^2}{n+L}}^{\infty}\frac{t^{\frac{L-2}{2}}}{(1+t)^{\frac{n+L-1}{2}}}dt$$
$$\sim\frac{\Gamma\big(\frac{1}{2}r_j^2\frac{L}{2}\big)}{\Gamma\big(\frac{L}{2}\big)}. \tag{B.2.22}$$

Additionally:

$$\frac{1}{B\big(\frac{n}{2},\frac{L}{2}\big)}\sim\frac{\big(\frac{n-1}{2}\big)^{\frac{L+1}{2}}}{\Gamma\big(\frac{L+1}{2}\big)}. \tag{B.2.23}$$

Using the gamma doubling formula all in all gives:

$$t^{\text{IndSpherical}}\Big(r_j,s_j\Big)\sim\frac{1}{\sqrt{2\pi}}2^{\frac{L-2}{2}}s_j^L\,\mathrm{e}^{-\frac{1}{2}s^2}\,\frac{\Gamma\big(\frac{1}{2}r_j^2\frac{L}{2}\big)}{\Gamma(L)}. \tag{B.2.24}$$

□

Theorem B.2.4 (Almost square and weak spherical component)**.** *Let A be a matrix pertaining to the real induced spherical ensemble, in the regime of almost square matrices with weak spherical component: $L=O(1)$ and $n-N=O(1)$. In addition let $u\in\mathbb{R}$ such that $0<|u|<\infty$ and let $r_1,\dots,r_j\in\mathbb{R}$ as well as $s_1,\dots,s_m\in\mathbb{C}_+\backslash\mathbb{R}$. Furthermore set $x_j=u+\frac{r_j}{\sqrt{n+L}}$ for $j=1,\dots,K'$, $z_m=u+\frac{s_m}{\sqrt{n+L}}$ for $m=1,\dots,L'$, then:*

$$\lim_{N\to\infty}\frac{1}{(n+L)^N}\,R^{\mathit{IndSpherical}}_{K',L'}(x_1,\dots,x_{K'},z_1,\dots,z_{L'})$$
$$=\mathrm{Pfaff}\begin{bmatrix}K^{\mathit{Generalized}}(r_j,r_{j'}) & K^{\mathit{Generalized}}(r_j,s_{m'})\\ K^{\mathit{Generalized}}(s_m,r_{j'}) & K^{\mathit{Generalized}}(s_m,s_{m'})\end{bmatrix}, \tag{B.2.25}$$

with $j,j'=1,\dots,K'$, $m,m'=1,\dots,L'$ and:

$$\rho(u):=\frac{1}{\pi}\frac{1}{(1+u^2)^2}. \tag{B.2.26}$$

At the origin $u=0$:

$$\lim_{N\to\infty}\frac{1}{(n+L)^N}\,R^{\mathit{IndSpherical}}_{K',L'}(x_1,\dots,x_{K'},z_1,\dots,z_{L'})$$
$$=\mathrm{Pfaff}\begin{bmatrix}K^{\mathit{Generalized}}_{\mathit{origin}}(r_j,r_{j'}) & K^{\mathit{Generalized}}_{\mathit{origin}}(r_j,s_{m'})\\ K^{\mathit{Generalized}}_{\mathit{origin}}(s_m,r_{j'}) & K^{\mathit{Generalized}}_{\mathit{origin}}(s_m,s_{m'})\end{bmatrix}. \tag{B.2.27}$$

B.3 Correlation function asymptotics for the real induced Jacobi ensemble

Theorem B.3.1 (Strong rectangularity and strong non-orthogonality)**.** *Let A be a matrix pertaining to the real induced Jacobi ensemble, in the regime of strong rectangularity and strong non-orthogonality: $L = N\alpha$ and $K = kN$. Set $\frac{L}{l_N} := \mu_1$ and $\frac{M}{K} =: \mu_2$. In addition let $u \in \mathbb{R}$ such that $\sqrt{\mu_1} < |u| < \sqrt{\mu_2}$ and let $r_1, \ldots, r_j \in \mathbb{R}$ as well as $s_1, \ldots, s_m \in \mathbb{C}_+ \backslash \mathbb{R}$. Furthermore set $x_j = u + \frac{r_j}{\sqrt{l_M}}$ for $j = 1, \ldots, K'$, $z_m = u + \frac{s_m}{\sqrt{l_M}}$ for $m = 1, \ldots, L'$, then:*

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{l_M^N} R_{K',L'}^{IndJacobi}(x_1, \ldots, x_{K'}, z_1, \ldots, z_{L'})\\ &= \mathrm{Pfaff}\begin{bmatrix} K^{Generalized}(r_j, r_{j'}) & K^{Generalized}(r_j, s_{m'}) \\ K^{Generalized}(s_m, r_{j'}) & K^{Generalized}(s_m, s_{m'}) \end{bmatrix},\end{aligned} \tag{B.3.1}$$

with $j, j' = 1, \ldots, K'$, $m, m' = 1, \ldots, L'$ and:

$$\rho(u) := \frac{1}{\pi} \frac{1}{(1-u^2)^2} \big[\Theta(|u| - \sqrt{\mu_1}) - \Theta(|u| - \sqrt{\mu_2})\big]. \tag{B.3.2}$$

Theorem B.3.2 (Almost square and strong non-orthogonality)**.** *Let A be a matrix pertaining to the real induced Jacobi ensemble in the regime of almost square matrices with strong non-orthogonality: $L = O(1)$ and $K = kN$. Set $\frac{1}{K} =: \mu_2$. In addition let $u \in \mathbb{R}$ such that $0 < |u| < \sqrt{\mu_2}$ and let $r_1, \ldots, r_j \in \mathbb{R}$ as well as $s_1, \ldots, s_m \in \mathbb{C}_+ \backslash \mathbb{R}$. Furthermore set $x_j = u + \frac{r_j}{\sqrt{l_M}}$ for $j = 1, \ldots, K'$, $z_m = u + \frac{s_m}{\sqrt{l_M}}$ for $m = 1, \ldots, L'$, then:*

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{l_M^N} R_{K',L'}^{IndJacobi}(x_1, \ldots, x_{K'}, z_1, \ldots, z_{L'})\\ &= \mathrm{Pfaff}\begin{bmatrix} K^{Generalized}(r_j, r_{j'}) & K^{Generalized}(r_j, s_{m'}) \\ K^{Generalized}(s_m, r_{j'}) & K^{Generalized}(s_m, s_{m'}) \end{bmatrix},\end{aligned} \tag{B.3.3}$$

with $j, j' = 1, \ldots, K'$, $m, m' = 1, \ldots, L'$ and:

$$\rho(u) := \frac{1}{\pi} \frac{1}{(1-u^2)^2} \Theta(\sqrt{\mu_2} - |u|). \tag{B.3.4}$$

At the origin:

$$\lim_{N\to\infty} \frac{1}{l_M^N} R_{K',L'}^{IndJacobi}(x_1, \ldots, x_{K'}, z_1, \ldots, z_{L'})$$
$$= \text{Pfaff} \begin{bmatrix} K_{origin}^{Generalized}(r_j, r_{j'}) & K_{origin}^{Generalized}(r_j, s_{m'}) \\ K_{origin}^{Generalized}(s_m, r_{j'}) & K_{origin}^{Generalized}(s_m, s_{m'}) \end{bmatrix}. \qquad \text{(B.3.5)}$$

Appendix C

Pfaffian kernel entries

C.1 Proof of theorem 4.1.18

Proof. Equipped with the appropriate skew-orthogonal polynomials and their normalisation the task of determining the entries of the matrix kernel for the $(K', L')-$correlation functions can now proceed. First equation (4.1.59) implies for $w, z \in \mathbb{C}$:

$$DS_N^{\mathrm{IndGin}}(z,v) = \frac{1}{\sqrt{2\pi}} w_{\mathrm{IndGin},1}(v) w_{\mathrm{IndGin},1}(z) \sum_{j=0}^{N-2} \frac{(vz)^j}{\Gamma(L+j+1)}. \tag{C.1.1}$$

Noting that $S_N^{\mathrm{IndGin}}(z,v) = iDS_N^{\mathrm{IndGin}}(z,\bar{v})$ and $IS_N^{\mathrm{IndGin}}(z,v) = -DS_N^{\mathrm{IndGin}}(\bar{z},\bar{v})$ we have completely determined the entries of the complex-complex matrix kernel.

Let us next consider the case $x \in \mathbb{R}$, $z \in \mathbb{C}$. The following approach is borrowed from [FN08]. We observe that:

$$q_{2j+1}^{\mathrm{IndGin}}(x) = -\exp\Big(\frac{1}{2}x^2\Big)x^{-L}\frac{\partial}{\partial x}\left[\exp\Big(-\frac{1}{2}x^2\Big)x^{2j+L}\right], \tag{C.1.2}$$

which implies for $j > 0$:

$$\tau_{2j+1}(x) = \exp\Big(-\frac{1}{2}x^2\Big)x^{j+L}. \tag{C.1.3}$$

Furthermore direct computation shows that:

$$\tau_1(x) - \frac{L}{2}\int_{\mathbb{R}} \mathrm{sgn}(x-t)\exp\Big(-\frac{1}{2}t^2\Big)x^{L-1}dt = \exp\Big(-\frac{1}{2}x^2\Big)x^{2j+L}. \tag{C.1.4}$$

All in all:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\text{IndGin}}} \tilde{q}_{2j}^{\text{IndGin}}(z)\tau_{2j+1}(x) = \frac{1}{\sqrt{2\pi}} w_{\text{IndGin},1}(z) e^{-\frac{1}{2}x^2} \sum_{j=1}^{\frac{N}{2}-1} \frac{(xz)^{L+2j}}{\Gamma(L+2j+1)}$$
$$+ \frac{1}{\sqrt{2\pi}\Gamma(L+1)} w_{\text{IndGin},1}(z) z^L \left[\tau_1(x) - \frac{L}{2}\int_{\mathbb{R}} \text{sgn}(x-t) e^{-\frac{1}{2}t^2} x^{L-1} dt\right]$$
$$+ \frac{1}{\sqrt{2\pi}\Gamma(L+1)} w_{\text{IndGin},1}(z) z^L \frac{L}{2}\int_{\mathbb{R}} \text{sgn}(x-t) e^{-\frac{1}{2}t^2} x^{L-1} dt$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} w_{\text{IndGin},1}(z) \sum_{j=0}^{\frac{N}{2}-1} \frac{(xz)^{L+2j}}{\Gamma(L+2j+1)} + \frac{1}{\sqrt{2\pi}} w_{\text{IndGin},1}(z) z^L 2^{\frac{L}{2}-1} \frac{\Gamma(\frac{L}{2}, \frac{1}{2}x^2)}{\Gamma(L+1)}.$$

In addition to that:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\text{IndGin}}} \tilde{q}_{2j+1}^{\text{IndGin}}(z)\tau_{2j}(x) = \frac{1}{\sqrt{2\pi}} w_{\text{IndGin},1}(z) \sum_{j=1}^{\frac{N}{2}-1} \frac{\left[z^{2j+1} - (L+2j)z^{2j-1}\right]\tau_{2j}(x)}{\Gamma(L+2j+1)}$$
$$+ \frac{1}{\sqrt{2\pi}\Gamma(L+1)} w_{\text{IndGin},1}(z)\tau_0(x). \tag{C.1.5}$$

Rearranging the summation gives:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\text{IndGin}}} \tilde{q}_{2j+1}^{\text{IndGin}}(z)\tau_{2j}(x) = \frac{1}{\sqrt{2\pi}\Gamma(L+N-1)} w_{\text{IndGin},1}(z) z^{N-1}\tau_{N-2}(x)$$
$$- \frac{1}{\sqrt{2\pi}} w_{\text{IndGin},1}(z) \sum_{j=0}^{\frac{N}{2}-2} \frac{\left[\tau_{2j+2}(x) - (L+j+1)\tau_{2j}(x)\right] z^{2j+1}}{\Gamma(L+2j+1)}. \tag{C.1.6}$$

Another differential equation:

$$q_{2j+2}^{\text{IndGin}}(x) - (2j+L+1) q_{2j}^{\text{IndGin}}(x) = -e^{\frac{1}{2}x^2} x^{-L} \frac{\partial}{\partial x}\left[\exp\Big(-\frac{1}{2}x^2\Big) x^{2j+L+1}\right],$$

leads to:

$$\tau_{2j+2}(x) - (2j+L+1)\tau_{2j}(x) = \mathrm{e}^{-\frac{1}{2}x^2}\, x^{2j+L+1}. \tag{C.1.7}$$

As a consequence:

$$
\begin{aligned}
&\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\mathrm{IndGin}}} \tilde{q}_{2j+1}^{\mathrm{IndGin}}(z)\tau_{2j}(x) \\
= &- \frac{1}{\sqrt{2\pi}} w_{\mathrm{IndGin},1}(z) z^{N-1} 2^{\frac{L}{2}+N-\frac{3}{2}} \operatorname{sgn}(x) \frac{\gamma(\frac{L}{2}+\frac{N}{2}-\frac{1}{2}, \frac{1}{2}x^2)}{\Gamma(L+N-1)} \\
&- \frac{1}{\sqrt{2\pi}} w_{\mathrm{IndGin},1}(z) e^{-\frac{1}{2}x^2} \sum_{j=0}^{\frac{N}{2}-2} \frac{(xz)^{2j+1}}{\Gamma(L+2j+2)}.
\end{aligned} \tag{C.1.8}
$$

Finally we obtain:

$$
\begin{aligned}
S_N^{\mathrm{IndGin}}(z,x) = &\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} w_{\mathrm{IndGin},1}(z) \sum_{j=0}^{N-2} \frac{(xz)^{2j}}{\Gamma(L+2j+1)} \\
&+ \frac{1}{\sqrt{2\pi}} w_{\mathrm{IndGin},1}(z) z^{N-1} 2^{\frac{L}{2}+N-\frac{3}{2}} \operatorname{sgn}(x) \frac{\gamma(\frac{L}{2}+\frac{N}{2}-\frac{1}{2}, \frac{1}{2}x^2)}{\Gamma(L+N-1)} \\
&+ \frac{1}{\sqrt{2\pi}} w_{\mathrm{IndGin},1}(z) 2^{\frac{L}{2}-1} \frac{\Gamma(\frac{L}{2}, \frac{1}{2}x^2)}{\Gamma(L+1)}.
\end{aligned} \tag{C.1.9}
$$

The last entry that requires explicit computation is $IS_N^{\mathrm{IndGin}}(x,y)$ for $x,y \in \mathbb{R}$. Here the relationship:

$$
IS_N^{\mathrm{IndGin}}(x,y) = -\int_x^y S_N^{\mathrm{IndGin}}(t,y)dt \tag{C.1.10}
$$

comes handy. Using the expression obtained for $S_N^{\mathrm{IndGin}}(x,y)$ and in addition to that employing the integral representation:

$$
e^{-ty} \sum_{j=0}^{N-2} \frac{(ty)^{L+2j}}{\Gamma(L+2j+1)} = \left[\frac{\gamma(L,ty)}{\Gamma(L)} - \frac{\gamma(L+N-1,ty)}{\Gamma(L+N-1)} \right] \tag{C.1.11}
$$

lead to the following starting point for our derivation:

$$
\begin{aligned}
IS_N^{\mathrm{IndGin}}(x,y) = &-\frac{1}{\sqrt{2\pi}\Gamma(L)} \int_x^y e^{-\frac{1}{2}(t-y)^2} (y-t)\gamma(L,ty)dt \\
&+ \frac{1}{\sqrt{2\pi}\Gamma(L+N-1)} \int_x^y e^{-\frac{1}{2}(t-y)^2} (y-t)\gamma(L+N-1,ty)dt \\
&- \frac{1}{\sqrt{2\pi}} \operatorname{sgn}(x) 2^{\frac{L}{2}+N-\frac{3}{2}} \frac{\gamma(L+\frac{N}{2}-\frac{1}{2}, \frac{1}{2}x^2)}{\Gamma(L+N-1)} \int_x^y e^{\frac{1}{2}t^2} t^{L+N-1} dt \\
&- \frac{1}{\sqrt{2\pi}} \frac{\Gamma(\frac{L}{2}, \frac{1}{2}x^2)}{\Gamma(L+1)} \int_x^y e^{\frac{1}{2}t^2} t^L dt.
\end{aligned} \tag{C.1.12}
$$

The above expression can be simplified by employing integration by parts with respect to t. As a conclusion we have derived all the possible entries of the Pfaffian matrix kernel. $\square$

C.2 Proof of theorem 4.2.12

Proof. Equipped with the appropriate skew-orthogonal polynomials, the task of determining the entries of the Pfaffian kernel can now proceed. First note that (4.2.99) and (4.2.100) give:

$$DS_N^I(z,v) = (z-v)w^I(z)w^I(v)\sum_{j=0}^{N-2} S^I(N-2,j)(zv)^j. \tag{C.2.1}$$

for all $z, v \in \mathbb{C}$ with:

$$\begin{aligned}
\sum_{j=0}^{N-2} S^{\text{IndJacobi}}(N-2,j)(zv)^j &= \frac{2}{\pi}\sum_{j=0}^{N-2}\frac{\Gamma(K-N+j+1)}{\Gamma(L+j+1)\Gamma(l_M-1)}(vz)^j \qquad \text{(C.2.2)}\\
\sum_{j=0}^{N-2} S^{\text{IndSpherical})}(N-2,j)(zv)^j &= \frac{2}{\pi}\sum_{j=0}^{N-2}\frac{\Gamma(n+N+1)}{\Gamma(L+j+1)\Gamma(n-j-1)}(vz)^j.
\end{aligned}$$

From the definitions of τ_j and the kernel entries it follows, that for $z, v \in \mathbb{C}\backslash\mathbb{R}$:

$$S_N^I(z,v) = iDS_N^I(z,\bar{v}) \qquad IS_N^I(z,v) = -DS_N^I(\bar{z},\bar{v}). \tag{C.2.3}$$

Furthermore:

$$S_N^I(x,z) = iDS_N^I(x,\bar{z}) \quad \text{for} \quad x \in \mathbb{R}\,, z \in \mathbb{C}\backslash\mathbb{R}. \tag{C.2.4}$$

The next entry to compute is $S_N^I(x,y)$ for $x, y \in \mathbb{R}$. This is the most involved calculation. First note the following equality for $j > 0$:

$$\begin{aligned}
q_{2j+1}^{\text{IndJacobi}}(x) &= -\frac{1}{l_N+2j}w_{\text{IndJacobi},1}^{-1}(x)\times\\
&\quad \frac{d}{dt}\left(w_{\text{IndJacobi},1}(t)\,q_{2j}^{\text{IndJacobi}}(t)(1-t^2)\right) \qquad \text{(C.2.5)}\\
q_{2j+1}^{\text{IndSpherical}}(x) &= -\frac{1}{n-2j-1}w_{\text{IndSpherical},1}^{-1}(x)\times\\
&\quad \frac{d}{dt}\left(w_{\text{IndSpherical},1}(t)\,q_{2j}^{\text{IndSpherical}}(t)(1+t^2)\right), \qquad \text{(C.2.6)}
\end{aligned}$$

which can easily be checked through differentiation. This implies for $j > 0$:

$$\begin{aligned}
\tau_{2j+1}^{\text{IndJacobi}}(x) &= -\frac{1}{2}\int_{-1}^{1} \operatorname{sgn}(x-t) - \frac{1}{l_N+2j} w_{\text{IndJacobi},1}^{-1}(x) \times \\
&\quad \frac{d}{dt}\left(w_{\text{IndJacobi},1}(t) q_{2j}^{\text{IndJacobi}}(t)(1-t^2)\right) w_{\text{IndJacobi},1}(t)dt \\
&= \frac{1}{2}\frac{1}{l_N+2j}\int_{-1}^{1} \operatorname{sgn}(x-t)\frac{d}{dt}\left(w_{\text{IndJacobi},1}(t) q_{2j}^{\text{IndJacobi}}(t)(1-t^2)\right) dt \\
&= \frac{1}{2\sqrt{2}}\frac{\sqrt{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}}{l_N+2j}\Big[-\int_{x}^{1} \frac{d}{dt}\left(w_{\text{IndJacobi},1}(t) q_{2j}^{\text{IndJacobi}}(t)(1-t^2)\right) dt \\
&\quad + \int_{-1}^{x} \frac{d}{dt}\left(w_{\text{IndJacobi},1}(t) q_{2j}^{\text{IndJacobi}}(t)(1-t^2)\right) dt\Big] \\
&= \frac{\sqrt{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}}{l_N+2j} x^{L+2j}(1-x^2)^{\frac{l_M}{2}}. \qquad \text{(C.2.7)}
\end{aligned}$$

Similarly:

$$\tau_{2j+1}^{\text{IndSpherical}}(x) = \frac{\sqrt{B\left(\frac{1}{2},\frac{n+L+1}{2}\right)}}{n-2j-1} x^{L+2j}(1+x^2)^{\frac{-n-L+1}{2}}. \qquad \text{(C.2.8)}$$

We first compute:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_{2j}^{I}}\, \tilde{q}_{2j}^{I}(x)\, \tau_{2j+1}(y), \qquad \text{(C.2.9)}$$

giving:

$$\begin{aligned}
&\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_{2j}^{\text{IndJacobi}}}\, \tilde{q}_{2j}^{\text{IndJacobi}}(x)\, \tau_{2j+1}(y) = \frac{1}{2} w_{\text{IndJacobi},1}(x) w_{\text{IndJacobi},1}(y)(1-y^2) \times \\
&\sum_{j=1}^{\frac{N}{2}-1} \frac{\Gamma(L_N+2j)}{\Gamma(l_M+1)\Gamma(L+2j+1)}(xy)^{2j} + w_{\text{IndJacobi},1}(x)\frac{\Gamma(l_N+1)}{\Gamma(l_M+1)\Gamma(L+1)}\tau_1(y) \\
&\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_{2j}^{\text{IndSpherical}}}\, \tilde{q}_{2j}^{\text{IndSpherical}}(x)\, \tau_{2j+1}(y) = \frac{1}{2} w_{\text{IndSpherical},1}(x) w_{\text{IndSpherical},1}(y)(1+y^2) \times \\
&\sum_{j=1}^{\frac{N}{2}-1} \frac{\Gamma(n+L+1)}{\Gamma(N-2j)\Gamma(L+2j+1)}(xy)^{2j} + w_{\text{IndSpherical},1}(x)\frac{\Gamma(n+L+1)}{\Gamma(n-1)\Gamma(L+1)}\tau_1(y).
\end{aligned}$$

Furthermore note:

$$x - \frac{L}{l_N}\frac{1}{x} = -\frac{1}{l_N} w^{-1}_{\text{IndJacobi},1}(x)\frac{d}{dt}\left(w_{\text{IndJacobi},1}(t)(1-t^2)\right) \tag{C.2.10}$$

$$x - \frac{L}{n-1}\frac{1}{x} = -\frac{1}{n-1} w^{-1}_{\text{IndSpherical},1}(x)\frac{d}{dt}\left(w_{\text{IndSpherical},1}(t)(1+t^2)\right). \tag{C.2.11}$$

As a result:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r^{\text{IndJacobi}}_{2j}}\, \tilde{q}^{\text{IndJacobi}}_{2j}(x)\, \tau_{2j+1}(y) = \frac{1}{2\pi} w_{\text{IndJacobi},1}(x) w_{\text{IndJacobi},1}(y)(1-y^2) \times$$
$$\sum_{j=1}^{\frac{N}{2}-1} \frac{\Gamma(L_N+2j)}{\Gamma(l_M-1)\Gamma(L+2j+1)}(xy)^{2j} - \frac{1}{2\pi}\frac{\Gamma(l_N+1)}{\Gamma(l_M-1)\Gamma(L+1)} w_{\text{IndJacobi},1}(x) \times$$
$$\left(\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}\right)^{\frac{1}{2}} \int_{-1}^{1} \operatorname{sgn}(y-t) t^{L-1}(1-t^2)^{\frac{l_M-2}{2}} dt, \tag{C.2.12}$$

as well as:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r^{\text{IndSpherical}}_{2j}}\, \tilde{q}^{\text{IndSpherical}}_{2j}(x)\, \tau_{2j+1}(y) = \frac{1}{2\pi} w_{\text{IndSpherical},1}(x) w_{\text{IndSpherical},1}(y) \times$$
$$(1+y^2)\sum_{j=1}^{\frac{N}{2}-1} \frac{\Gamma(n+L+1)}{\Gamma(N-2j)\Gamma(L+2j+1)}(xy)^{2j} - \frac{1}{2\pi}\frac{\Gamma(n+L+1)}{\Gamma(n-1)\Gamma(L+1)} \times$$
$$w_{\text{IndSpherical},1}(x)\left(\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}\right)^{\frac{1}{2}} \int_{-1}^{1} \operatorname{sgn}(y-t) t^{L-1}(1+t^2)^{-\frac{n+L+1}{2}} dt. \tag{C.2.13}$$

In addition we need to compute:

$$\begin{aligned}
&-\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}} \int_{-1}^{1} \operatorname{sgn}(y-t) t^{L-1}(1-t^2)^{\frac{l_M-2}{2}} dt \\
=&-\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}} \left[-\int_{y}^{1} t^{L-1}(1-t^2)^{\frac{l_M-2}{2}} dt + \int_{-1}^{y} t^{L-1}(1-t^2)^{\frac{l_M-2}{2}} dt\right] \\
=&\sqrt{\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}} \int_{|y|}^{1} t^{L-1}(1-t^2)^{\frac{l_M-2}{2}} dt \\
=&\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}} \int_{y^2}^{1} t^{\frac{L}{2}-1}(1-t)^{\frac{l_M-2}{2}} dt \\
=&\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}{2}} B\left(\frac{L}{2},\frac{l_M}{2}\right) I_{1-y^2}\left(\frac{L}{2},\frac{l_M}{2}\right).
\end{aligned}$$

Similarly:

$$\begin{aligned}
&\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{n+L+1}{2}\right)}{2}}\int_{-1}^{1}\operatorname{sgn}(y-t)t^{L-1}(1+t^2)^{-\frac{n+L+1}{2}}dt\\
=&\frac{1}{2}\sqrt{\frac{B\left(\frac{1}{2},\frac{n+L+1}{2}\right)}{2}}B\Big(\frac{L}{2},\frac{n+1}{2}\Big)I_{\frac{1}{1+y^2}}\Big(\frac{L+2}{2},\frac{n-1}{2}\Big).
\end{aligned}$$

Then finally:

$$\begin{aligned}
&\sum_{j=0}^{\frac{N}{2}-1}\frac{1}{r_j^{\text{IndJacobi}}}\,\tilde{q}_{2j}^{\text{IndJacobi}}(x)\,\tau_{2j+1}(y)\\
=&\frac{1}{2\pi}w_{\text{IndJacobi},1}(x)w_{\text{IndJacobi},1}(y)(1-y^2)\sum_{j=1}^{\frac{N}{2}-1}\frac{\Gamma(L_N+2j)}{\Gamma(l_M-1)\Gamma(L+2j+1)}(xy)^{2j}\\
&+\frac{1}{B\left(\frac{l_M}{2},\frac{L+1}{2}\right)}x^L(1-x^2)^{\frac{l_M-2}{2}}I_{1-y^2}\Big(\frac{L}{2},\frac{l_M}{2}\Big)
\end{aligned}\tag{C.2.14}$$

and:

$$\begin{aligned}
&\sum_{j=0}^{\frac{N}{2}-1}\frac{1}{r_j^{\text{IndSpherical}}}\,\tilde{q}_{2j}^{\text{IndSpherical}}(x)\,\tau_{2j+1}(y)\\
=&\frac{1}{2\pi}w_{\text{IndSpherical},1}(x)w_{\text{IndSpherical},1}(y)(1+y^2)\sum_{j=1}^{\frac{N}{2}-1}\frac{\Gamma(n+L+1)}{\Gamma(N-2j)\Gamma(L+2j+1)}(xy)^{2j}\\
&+\frac{1}{B\left(\frac{n}{2},\frac{L+1}{2}\right)}x^L(1+x^2)^{-\frac{n+L+1}{2}}I_{\frac{1}{1+y^2}}\Big(\frac{L+2}{2},\frac{n-1}{2}\Big)
\end{aligned}\tag{C.2.15}$$

In addition we need to compute:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r^I_{2j}}\, \tilde{q}^I_{2j+1}(x)\, \tau_{2j}(y) \tag{C.2.16}$$

$$\begin{aligned}
=&w_I(x)\Big\{\frac{1}{r^I_0}x\tau_0(y) + \sum_{j=1}^{\frac{N}{2}-1} \frac{1}{r^I_{2j}}\Big(x^{2j+1} - \frac{S^I(2j,2j-1)}{S^I(2j,2j)}x^{2j-1}\Big)\tau_{2j}(y)\Big\}\\
=&w_I(x)\Big\{\frac{1}{r^I_0}x\tau_0(y) - \frac{1}{r^I_2}\frac{S^I(2,1)}{S^I(2,2)}x^1\tau_2(y) + \frac{1}{r^I_2}x^3\tau_2(y)\\
&- \frac{1}{r^I_4}\frac{S^I(4,3)}{S^I(4,4)}x^3\tau_4(y) + \frac{1}{r^I_4}x^5\tau_4(y) - \cdots - \frac{1}{r^I_{N-4}}\frac{S^I(N-4,N-5)}{S^I(N-4,N-4)}x^{N-5}\tau_{N-4}(y)\\
&+ \frac{1}{r^I_{N-4}}x^{N-3}\tau_{N-4}(y) - \frac{1}{r^I_{N-2}}\frac{S^I(N-2,N-3)}{S^I(N-2,N-2)}x^{N-3}\tau_{N-2}(y)\\
&+ \frac{1}{r^I_{N-2}}x^{N-1}\tau_{2N-2}(y)\Big\} = w_{I,1}(x)\Big\{\frac{1}{r^I_{N-2}}x^{N-1}\tau_{2N-2}(y)\\
&- \sum_{j=1}^{\frac{N}{2}-2} \frac{1}{r^I_{2j+1}}x^{2j+1}\Big(\tau_{2j+2}(y) - \frac{S^I(2j+1,2j)}{S^I(2j+1,2j+1)}\tau_{2j}(y)\Big)\Big\}
\end{aligned}$$

Moreover:

$$\begin{aligned}
&q^{\text{IndJacobi}}_{2j+2}(x) - \frac{S^{\text{IndJacobi}}(2j+1,2j)}{S^{\text{IndJacobi}}(2j+1,2j+1)}q^{\text{IndJacobi})}_{2j}(x)\\
=&-\frac{1}{l_N+2j+1}w^{-1}_{\text{IndJacobi},1}(x)\frac{d}{dt}\left(w_{\text{IndJacobi},1}(t)q^{\text{IndJacobi}}_{2j+1}(t)(1-t^2)\right) \qquad \text{(C.2.17)}\\
&q^{\text{IndSpherical}}_{2j+2}(x) - \frac{S^{\text{IndSpherical}}(2j+1,2j)}{S^{\text{IndSpherical}}(2j+1,2j+1)}q^{\text{IndSpherical}}_{2j}(x)\\
=&-\frac{1}{n-2j-2}w^{-1}_{\text{IndSpherical},1}(x)\frac{d}{dt}\left(w_{\text{IndSpherical},1}(t)q^{\text{IndSpherical}}_{2j+1}(t)(1+t^2)\right),
\end{aligned} \tag{C.2.18}$$

which implies:

$$\tau_{2j+2}(y) - \frac{S^{\text{IndJacobi}}(2j+1,2j)}{S^{\text{IndJacobi}}(2j+1,2j+1)}\tau_{2j}(y) = \frac{\sqrt{B\left(\frac{1}{2},\frac{l_M-1}{2}\right)}}{l_N+2j+1}x^{L+2j+1}(1-x^2)^{\frac{l_M}{2}}$$

$$\tau_{2j+2}(y) - \frac{S^{\text{IndSpherical}}(2j+1,2j)}{S^{\text{IndSpherical}}(2j+1,2j+1)}\tau_{2j}(y) = \frac{\sqrt{B\left(\frac{1}{2},\frac{n+L+1}{2}\right)}}{n-2j-2}x^{L+2j+1}(1+x^2)^{-\frac{n+L-1}{2}}.$$

As a consequence:

$$\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_{2j}^{\text{IndJacobi}}} \, \tilde{q}_{2j+1}^{\text{IndJacobi}}(x) \, \tau_{2j}(y) = -\frac{l_M(l_M-1)}{2\pi} w_{\text{IndJacobi},1}(x) w_{\text{IndJacobi},1}(y) \times$$

$$(1-y^2) \sum_{j=0}^{\frac{N}{2}-2} \frac{\Gamma(L_N+2j+1)}{\Gamma(l_M+1)\Gamma(L+2j+2)} (xy)^{2j+1}$$

$$+ \frac{1}{\pi} \frac{\Gamma(N-1)}{\Gamma(l_M-1)\Gamma(M-1)} w_{\text{IndJacobi},1}(x) x^{N-1} \tau_{N-2}(y),$$

as well as:

$$\sum_{j=0}^{\frac{N}{2}-2} \frac{1}{r_{2j}^{\text{IndSpherical}}} \, \tilde{q}_{2j}^{\text{IndSpherical}}(x) \, \tau_{2j+1}^{\text{IndSpherical}}(y) = -\frac{1}{2\pi} w_{\text{IndSpherical},1}(x) w_{\text{IndSpherical},1}(y) \times$$

$$(1+y^2) \sum_{j=0}^{\frac{N}{2}-1} \frac{\Gamma(n+L+1)}{\Gamma(N-2j-1)\Gamma(L+2j+2)} (xy)^{2j+1}$$

$$+ \frac{1}{\pi} \frac{\Gamma(n+L+1)}{\Gamma(n-N+1)\Gamma(M-1)} w_{\text{IndSpherical},1}(x) x^{N-1} \tau_{N-2}(y).$$

Then finally:

$$\begin{aligned} &\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\text{IndJacobi}}} \, \tilde{q}_{2j+1}^{\text{IndJacobi}}(x) \, \tau_{2j}(y) \\ &= -\frac{l_M(l_M-1)}{2\pi} w_{\text{IndJacobi},1}(x) w_{\text{IndJacobi},1}(y)(1-y^2) \sum_{j=0}^{\frac{N}{2}-2} \frac{\Gamma(L_N+2j+1)}{\Gamma(l_M+1)\Gamma(L+2j+2)} (xy)^{2j+1} \\ &\quad - \frac{1}{B\left(\frac{l_M}{2}, \frac{M}{2}\right)} x^{M-1}(1-x^2)^{\frac{l_M-2}{2}} \operatorname{sgn}(y) I_{y^2}\left(\frac{M-1}{2}, \frac{l_M}{2}\right) \end{aligned} \tag{C.2.19}$$

$$\begin{aligned} &\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j^{\text{IndSpherical}}} \, \tilde{q}_{2j}^{\text{IndSpherical}}(x) \, \tau_{2j+1}(y) \\ &= -\frac{1}{2\pi} w_{\text{IndSpherical},1}(x) w_{\text{IndSpherical},1}(y)(1+y^2) \sum_{j=0}^{\frac{N}{2}-1} \frac{\Gamma(n+L+1)}{\Gamma(N-2j)\Gamma(L+2j+1)} (xy)^{2j} \\ &\quad - \frac{1}{B\left(\frac{M}{2}, \frac{n-N+1}{2}\right)} x^{L}(1+x^2)^{-\frac{n+L+1}{2}} \operatorname{sgn}(y) I_{\frac{y^2}{1+y^2}}\left(\frac{M+1}{2}, \frac{n-N+2}{2}\right). \end{aligned} \tag{C.2.20}$$

This gives the kernel entry $S^I(x,y)$. All remaining entries can easily be derived from $S^I(x,y)$. □

Appendix D

Miscellaneous

D.1 The Selberg integral and some related integrals

Theorem D.1.1 (Selberg's integral, [Meh04], page 309)**.** *Let $N > 1$ be a positive integer, furthermore let:*

$$\Delta(x_1, \dots, x_N) := \prod_{1 \le i < j \le N} (x_i - x_j) \tag{D.1.1}$$

and

$$\Phi(x_1, \dots, x_N) = \big|\Delta(x_1, \dots, x_N)\big|^{2\gamma} \prod_{j=1}^{N} x_j^{\alpha-1}(1-x_j)^{\beta-1} \tag{D.1.2}$$

Then:

$$\begin{aligned} I(\alpha, \beta, \gamma, N) &:= \int_0^1 \cdots \int_0^1 \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma\big(\alpha+\beta+(N+j-1)\gamma\big)} \end{aligned} \tag{D.1.3}$$

and for $1 \le m \le N$:

$$\begin{aligned} I(\alpha, \beta, \gamma, N, m) &:= \int_0^1 \cdots \int_0^1 x_1 \cdots x_m \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N \\ &= \prod_{j=1}^{m} \frac{\alpha+(N-j)\gamma}{\alpha+\beta+(2N-j-1)\gamma} I(\alpha, \beta, \gamma, N) \end{aligned} \tag{D.1.4}$$

valid for integer N and complex α, β with:

$$\mathrm{Re}(\alpha) > 0, \quad \mathrm{Re}(\beta) > 0, \quad \mathrm{Re}(\gamma) > -\min\big(\frac{1}{N}, \frac{\mathrm{Re}(\alpha)}{N-1}, \frac{\mathrm{Re}(\beta)}{N-1}\big). \tag{D.1.5}$$

A consequence of Selberg's and Aomoto's integral is:

Corollary D.1.2 ([Meh04], page 310)**.** *Let $\Phi(x_1, \dots, x_N)$ be defined as in (D.1.1), then:*

$$\begin{aligned} B(m_1, m_2) &:= \int_0^1 \cdots \int_0^1 \prod_{i_1=1}^{m_1} x_{i_1} \prod_{i_2=m_1+1}^{m_1+m_2} (1 - x_{i_2}) \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N \\ &= \prod_{j=1}^{m} \frac{\prod_{i_1=1}^{m_1} (\alpha + (N - i_1)\gamma) \prod_{i_2=1}^{m_2} (\beta + (N - i_2)\gamma)}{\prod_{i=1}^{m_1+m_2} (\alpha + \beta + (2N - i - 1)\gamma)} I(\alpha, \beta, \gamma, N). \end{aligned} \tag{D.1.6}$$

Lemma D.1.3 ([Meh04])**.** *Let $\Delta(x_1, \dots, x_N)$ as in (D.1.1) and define:*

$$\tilde{\Phi}(x_1, \dots, x_N) := \big|\Delta(x_1, \dots, x_N)\big|^{2\gamma} \prod_{j=1}^{N} x_j^{\alpha - \frac{1}{2}} \, \mathrm{e}^{-\frac{1}{2} x_j} \, . \tag{D.1.7}$$

Then:

$$\begin{aligned} J(\alpha, \gamma, N) &:= \int_0^\infty \cdots \int_0^\infty \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N \\ &= 2^{\frac{1}{2}N^2 + N\alpha} \prod_{j=0}^{N-1} \frac{\Gamma(\frac{3}{2} + \frac{1}{2}j)\Gamma(\alpha + \frac{1}{2} + \frac{1}{2}j)}{\Gamma(\frac{3}{2})}. \end{aligned} \tag{D.1.8}$$

D.2 Pfaffians

Let $A = (a_{ij})$ be a skew-symmetric matrix of even dimension $2N$. The Pfaffian of the matrix A is then defined as follows:

$$\mathrm{Pfaff}(A) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(2j-1)\sigma(2j)}, \tag{D.2.1}$$

where S_{2N} is the symmetric group of size $2N$ and $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation. Alternatively the Pfaffian is given by the square root of the determinant.

$$\mathrm{Pfaff}^2(A) = \det(A). \tag{D.2.2}$$

This is a classical result first proved by Thomas Muir. The Pfaffian of a skew-symmetric matrix of odd dimensions is zero. The following identities for the

Pfaffian hold:

$$\mathrm{Pfaff}(A^T) = (-1)^N \,\mathrm{Pfaff}(A) \tag{D.2.3}$$

$$\mathrm{Pfaff}(\alpha A) = \alpha^N \,\mathrm{Pfaff}(A). \tag{D.2.4}$$

Additionally for an arbitrary matrix B of dimension $2N \times 2N$:

$$\mathrm{Pfaff}(BAB^T) = \det(B)\,\mathrm{Pfaff}(A). \tag{D.2.5}$$

D.3 Proof of theorem 4.2.9 (a) for $K < M + N$

Proof. We need to determine $\langle \epsilon_k(A^T A)\rangle_A$ in the case that A is an induced Jacobi matrix with parameters K, M, N and $K < N + M$. Let

$$U = \begin{bmatrix} Q & B \\ P & D \end{bmatrix} \in \mathbb{R}^{K\times K} \tag{D.3.1}$$

a random orthogonal matrix. In addition let $Q \in \mathbb{R}^{M\times N}$ denote the top left corner of the matrix U. Note that due to the relation:

$$\det\left(zQ^TQ + I_N\right) = \sum_{j=0}^{N} z^{N-j}\epsilon_j(Q^TQ), \tag{D.3.2}$$

The function:

$$F(z) := \left\langle \det\left(zQ^TQ + I_N\right)\right\rangle_{O(N)} \tag{D.3.3}$$

is a generating function for $\langle\epsilon_k(Q^TQ)\rangle_A\rangle_{O(N)}$. We thus proceed to derive $F(z)$ by introducing two integration formulae:

1. Let $X \in \mathbb{R}^{M\times N}$ with $M \geq N$ be a "standing" rectangular random matrix with quadratization:

$$X = W\begin{bmatrix} G \\ 0 \end{bmatrix}, \tag{D.3.4}$$

see also equation (2.1.2). Then from theorem 2.2.1 for a suitable function f:

$$\int_{(X)} f(X^TX)(dX) \propto \int_{(G)} \det(G^TG)^{\frac{M-N}{2}} f(G^TG)(dG), \tag{D.3.5}$$

see also proof of theorem 4.2.7.

2. Let $X \in \mathbb{R}^{M\times N}$ with $M < N$ be a "laying" rectangular random matrix with

quadratization:

$$X^T = W^T \begin{bmatrix} G^T \\ 0 \end{bmatrix}, \tag{D.3.6}$$

Then:

$$\begin{aligned} &\int_{(X)} f(X^T X)(dX) \\ \propto &\int_{(W)} \int_{(G)} \det(G^T G)^{\frac{N-M}{2}} f\big(W^T \begin{bmatrix} G^T G & 0 \\ 0 & 0 \end{bmatrix} W\big)(dG)(W^T dW), \end{aligned} \tag{D.3.7}$$

Equipped with these integral formulae we note that:

$$F(z) \propto \int_{(Q)} \int_{(P)} \det\left(z Q^T Q + I_N\right) \delta\big(Q^T Q + P^T P - I_N\big)(P^T dP)(Q^T dQ) \tag{D.3.8}$$

due to the orthogonality of the matrix U. We now apply the quadratization procedure to the matrix Q:

$$Q = W \begin{bmatrix} G \\ 0 \end{bmatrix}, \tag{D.3.9}$$

as well as to the matrix P:

$$P = \begin{bmatrix} H, & 0 \end{bmatrix} V. \tag{D.3.10}$$

Here $H \in \mathbb{R}^{(K-M)\times(K-M)}$ and $V \in \mathbb{R}^{N\times N}$. □

As a result:

$$\begin{aligned} F(z) \propto \int_{(G)} \int_{(H)} &\det(G^T G)^{\frac{N-M}{2}} \det(H^T H)^{\frac{M+N-K}{2}} \det\left(z G^T G + I_N\right) \times \\ &\delta\big(G^T G + V^T \begin{bmatrix} H^T H & 0 \\ 0 & 0 \end{bmatrix} V I_N\big)(dG)(dH). \end{aligned} \tag{D.3.11}$$

Furthermore we can integrate out G, obtaining:

$$\begin{aligned} F(z) &\propto \int_{(H)} \det(H^T H)^{\frac{M+N-K-1}{2}} \det(I_N - H^T H)^{\frac{M-N}{2}} \times \\ &\qquad \det\Big(z\big(I_N - V^T \begin{bmatrix} H^T H & 0 \\ 0 & 0 \end{bmatrix} V\big) + I_N\Big)(dH) \qquad \text{(D.3.12)} \\ &\propto \int_{(H)} \det(H^T H)^{\frac{M+N-K-1}{2}} \det(I_N - H^T H)^{\frac{M-N}{2}} \times \\ &\qquad \det\Big(z \begin{bmatrix} I_{K-M} - H^T H & 0 \\ 0 & I_{M+N-K} \end{bmatrix} + I_N\Big)(dH). \qquad \text{(D.3.13)} \end{aligned}$$

As a consequence:

$$\langle \epsilon_j(Q^T Q) \rangle_Q = const. \Big\langle \epsilon_j \big(\begin{pmatrix} H^T H & 0 \\ 0 & I_{M+N-K} \end{pmatrix} \big) \Big\rangle_H. \qquad \text{(D.3.14)}$$

The normalization constant can be calculated by setting $x = 0$ in equation (D.3.13). Furthermore through simple combinatorics as well as the application of the Selberg integral from theorem D.1.1 show that the characteristic average over the real induced Jacobi ensemble takes the same expression for $K < N + M$ as for $K \geq N + M$.

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